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ARCH-COMP20 Category Report: Hybrid Systems Theorem Proving

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Abstract

This paper reports on the Hybrid Systems Theorem Proving (**HSTP**) category in the ARCH-COMP Friendly Competition 2020. The characteristic features of the HSTP category remain as in the previous editions [MST+18, MST+19]: *i)* The flexibility of programming languages as structuring principles for hybrid systems, *ii)* The unambiguity and precision of program semantics, and *iii)* The mathematical rigor of logical reasoning principles. The HSTP category especially features many nonlinear and parametric continuous and hybrid systems. Owing to the nature of theorem proving, HSTP again accommodates three modes: **A) Automatic** in which the entire verification is performed fully automatically without any additional input beyond the original hybrid system and its safety specification. **H) Hints** in which select proof hints are provided as part of the input problem specification, allowing users to communicate specific advice about the system such as loop invariants. **S) Scripted** in which a significant part of the verification is done with dedicated proof scripts or problem-specific proof tactics. This threefold split makes it possible to better identify the sources of scalability and efficiency bottlenecks in hybrid systems theorem proving. The existence of all three categories also makes it easier for new tools with a different focus to participate in the competition, wherever they focus on in the spectrum from fast proof checking all the way to full automation. The types of benchmarks considered and experimental findings with the participating theorem provers KeYmaera, KeYmaera X 4.6.3, KeYmaera X 4.8.0, Isabelle/HOL/Hybrid-Systems-VCs, and HHL Prover are described in this paper as well.

1 Introduction

This report summarizes the experimental results of the Hybrid Systems Theorem Proving (HSTP) category in the ARCH-COMP20 friendly competition. It is largely based on the previous editions of the HSTP category [MST+18, MST+19] and repeats benchmark and tool

descriptions in this report for convenience. The benchmark examples in the HSTP competition strive for a large variety in hybrid systems and games modeling patterns of basic extent to provide a low entry barrier for tools as well as examples at scale to identify opportunities for improving on proof automation, scalability and efficiency. The 214 examples in the benchmark competition are grouped into the following categories:

- Hybrid systems design shapes: small-scale examples over a large variety of model shapes to test for prover flexibility.
- Nonlinear continuous models: test for prover flexibility in terms of generating and proving properties about continuous dynamics.
- Hybrid games: small-scale examples with adversary dynamics in differential dynamic game logic.
- Hybrid systems case studies: hybrid systems models and specifications at scale to test for application scalability and efficiency.

In each of these categories, tools can select the degree of automation as follows, depending on their focus in the spectrum from fast proof checking to full proof automation:

(A) Automated: hybrid systems models and specifications are the only input, proofs and counterexamples are produced fully automatically.

(H) Hints: select proof hints (e.g., loop invariants) are provided as part of the specifications.

(S) Scripted: significant parts of the verification is done with dedicated problem-specific scripts or tactics.

All benchmark examples are available at https://github.com/LS-Lab/KeYmaeraX-projects/tree/master/benchmarks and specified in differential dynamic logic (dℒ) [Pla08, Pla17], whose format and ASCII syntax are presented in Section 2. The participating tools are presented in Section 3. An overview of the examples together with the findings from the competition is given in Section 4.

2 Problem Format

All benchmarks in the Hybrid Systems Theorem Proving (HSTP) category are written in differential dynamic logic (dℒ) [Pla08, Pla17] which has axioms and an unambiguous semantics available [BRV+17] in KeYmaera 3, KeYmaera X, Isabelle/HOL, and Coq. To make it easier for tools to participate in the HSTP category, almost all benchmarks in the HSTP category are differential dynamic logic formulas of the particular safety form

$$\phi \rightarrow [\alpha]\psi \tag{1}$$

where ϕ is a real arithmetic formula describing the initial conditions,
ψ is a real arithmetic formula describing the postcondition / set of safe states, and
α is the hybrid system described using hybrid programs as a program notation.

The dℒ formula (1) means that if the system starts in a state satisfying the initial condition ϕ, then all final states of all possible runs of the hybrid system α satisfy postcondition ψ. The operators / statements of hybrid programs are summarized in Table 1. Those of logical formulas in dℒ are summarized in Table 2. In particular, the hybrid program α contains both the discrete and continuous dynamics of the hybrid system.

An example with a purely continuous system is:

$$-\frac{4}{5} < x < -\frac{1}{3} \land -1 \leq y < 0 \;\rightarrow\; [x' = 2x - 2xy, y' = 2y - x^2 + y^2]\big(x + y \leq 1 \land (x \neq 0 \lor y \neq 0)\big) \tag{2}$$

Table 1: Statements of hybrid programs (Q is a first-order formula, α, β are hybrid programs)

Statement	Effect
$\alpha;\ \beta$	sequential composition where β starts after α finishes
$\alpha\ \cup\ \beta$	nondeterministic choice, following either alternative α or β
α^*	nondeterministic repetition, repeating α n times for any $n \in \mathbb{N}$
$x := \theta$	discrete assignment of the value of term θ to variable x (jump)
$x := *$	nondeterministic assignment of an arbitrary real number to x
$\big(x_1' = \theta_1, \ldots,$ $x_n' = \theta_n \& Q\big)$	continuous evolution of x_i along the differential equation system $x_i' = \theta_i$ of any non-negative duration restricted to remain in evolution domain Q throughout
$?Q$	test if formula Q holds at current state, abort program otherwise
$\texttt{if}(Q)\,\alpha$	run α if Q is true at current state, do nothing otherwise
$\texttt{if}(Q)\,\alpha\,\texttt{else}\,\beta$	run α if Q is true at current state, run β otherwise

Table 2: Operators of differential dynamic logic ($\mathsf{d}\mathcal{L}$) formulas

$\mathsf{d}\mathcal{L}$	Operator	Meaning
$\theta_1 \sim \theta_2$	comparison	true iff $\theta_1 \sim \theta_2$ with operator $\sim \in \{>, \geq, =, \neq, \leq, <\}$
$\neg\phi$	negation / not	true if ϕ is false
$\phi \wedge \psi$	conjunction / and	true if both ϕ and ψ are true
$\phi \vee \psi$	disjunction / or	true if ϕ is true or if ψ is true
$\phi \rightarrow \psi$	implication / implies	true if ϕ is false or ψ is true
$\phi \leftrightarrow \psi$	bi-implication / equivalent	true if ϕ and ψ are both true or both false
$\forall x\, \phi$	universal quantifier	true if ϕ is true for all values of variable x in $\mathbb{R}$
$\exists x\, \phi$	existential quantifier	true if ϕ is true for some values of variable x in $\mathbb{R}$
$[\alpha]\phi$	$[\cdot]$ modality / box	true if ϕ is true after all runs of hybrid program α
$\langle\alpha\rangle\phi$	$\langle\cdot\rangle$ modality / diamond	true if ϕ is true after at least one run of α

An example with a trivial hybrid system is:

$$v \geq 0 \wedge A > 0 \wedge b > 0 \rightarrow [\big(?v \leq 5; a := A \cup a := -b); \{x' = v, v' = a \,\&\, v \geq 0\}\big)^*]\, v \geq 0 \tag{3}$$

This particular example is completely trivial, because the postcondition $v \geq 0$ directly follows from the evolution domain constraint $v \geq 0$ in the differential equation. But safety properties become more exciting and more challenging when the postcondition is a different one. For example, $x \geq 10$ to say that the position is at least 10 always is much more complicated (and not even true for the above example).

Note that the operator precedence is such that unary operators bind stronger than binary operators and, just like in regular expressions, ; binds stronger than $\cup$. In particular, the controller in (3) is $(?(v \leq 5); a := A) \cup a := -b$.

ASCII syntax. The benchmark examples are specified in the $\mathsf{d}\mathcal{L}$ ASCII syntax and grouped into `.kyx` files, each containing several named archive entries. The ASCII syntax is a straightforward ASCII rendition of Tables 1 and 2, e.g., using `A->B` for $A \rightarrow B$ and using `A&B` for $A \wedge B$. The ASCII notation `alpha++beta` is used for $alpha \cup beta$. For improved readability in longer examples, braces {...} are used for grouping differential equation systems and other program operators. Like in C programs, assignments etc. end with explicit semicolons.

Archive entries follow the general shape below, listing optional definitions, system variables, a (safety) specification in $\mathtt{d}\mathcal{L}$, and optional tactic scripts. The example (3), specialized, just for the sake of illustration, to the case where $A = 5$, is written in ASCII KeYmaera X input as follows. Unlike the `ProgramVariables` and `Problem` block, the `Definitions` and `Tactic` blocks are optional. The symbols defined in the `Definitions` can be used in the `Problem` block or in other definitions. All examples are additionally provided in the format of the previous edition [MST+18] of the HSTP category.

```
ArchiveEntry "Benchmark Example 1"

Definitions                                  /* definitions cannot change their value */
  Real A = 5;                              /* real-valued maximum acceleration defined to be 5 */
  Real b;                                  /* real-valued braking, undefined so unknown value */
  Bool geq(Real x, Real y) <-> x>=y; /* predicate geq defined to be formula x>=y */
  HP drive ::= {                             /* program drive defined to choose either */
        ?v<=5; a:=A;                       /* maximum acceleration if slow enough */
    ++ a:=-b;                              /* or braking, nondeterministically */
  };
End.

ProgramVariables /* program variables may change their value over time */
  Real x;                                  /* real-valued position */
  Real v;                                  /* real-valued velocity */
  Real a;                                  /* current acceleration chosen by controller */
End.

Problem                                    /* conjecture in differential dynamic logic */
  v>=0 & A()>0 & b()>0                     /* initial condition */
  ->                                       /* implies */
  [                                        /* all runs of this hybrid program */
    {                                      /* braces {} group programs */
      drive;                               /* expand program drive here as defined above */
      { x'=v, v'=a & v>=0 }                /* differential equation system */
    }* @invariant(v>=0)                    /* loop repeats, with @invariant contract */
  ] v>=0                                   /* safety/postcondition after hybrid program */
End.

Tactic "Automated proof in KeYmaera X"
  master
End.

Tactic "Scripted proof in Bellerophon tactic language"
  implyR(1) ; loop("v>=0", 1) ; <(   /* < splits separate branches */
    id,                                    /* initial case: shown with close by identity */
    QE,                                      /* postcondition: prove by real arithmetic QE */
    /* induction step: decomposes hybrid program semi-explicitly */
    composeb(1) ; solve(1.1) ; choiceb(1) ; andR(1) ; <(   /* controller branches */
      composeb(1) ; testb(1) ; master,       /* decompose some steps then ask master */
      assignb(1) ; QE                        /* assignment, then real arithmetic */
   )
  )
End.

End. /* end of ArchiveEntry */
```

Background. A short survey on differential dynamic logic and hybrid programs can be found in a LICS'12 tutorial [Pla12a], a tutorial on its modeling principles in STTT [QML+16], a research monograph [Pla10b], and a comprehensive introduction in a textbook [Pla18]. The precise mathematical semantics of differential dynamic logic and its hybrid programs can be found in the literature as well, for example the most recent details in [Pla17], and a brief version in the LICS'12 tutorial [Pla12a].

3 Participating Tools

KeYmaera X. KeYmaera X [FMQ+15] is a theorem prover for the hybrid systems logic differential dynamic logic (dℒ). It implements the uniform substitution calculus of dℒ [Pla17].[1] KeYmaera X supports systems with nondeterministic discrete jumps, nonlinear differential equations, nondeterministic input, and it provides invariant construction and proving techniques for differential equations [SGJP16, PT18]. Unlike numerical hybrid systems reachability analysis tools, KeYmaera X also supports unbounded initial sets and unbounded time analysis. KeYmaera X participates in v4.6.3 (2019) and the latest v4.8.0 (2020).

Major improvements from KeYmaera X 4.6.3 to KeYmaera X 4.8.0 include a faster uniform substitution algorithm [Pla19], redesigned interaction with external tools, support for Wolfram Engine as a backend, builtin interval arithmetic tools, and numerous smaller stability and performance improvements.

KeYmaera X comes with automated proof search procedures that can be steered in the following ways: annotations in the input models provide additional design insight and, if available, are used to steer the invariant generation techniques in KeYmaera X; fine-grained control over proofs is available with proof scripts [FMBP17].

Extension with and experimentation in proof search without reducing trust in the prover is made possible on top of a small trusted kernel that checks all reasoning steps for soundness. The prover kernel contains a list of sound dℒ axioms that are instantiated using a uniform substitution proof rule [Pla17]. This approach isolates all soundness-critical reasoning in the prover kernel and obviates the intractable task of ensuring that each new proof search algorithm is implemented correctly. New proof search algorithms are always sound and can either be programmed directly in Scala (or Java) or can simply be added as a tactic in the hybrid systems tactic language Bellerophon [FMBP17].

The proof automation for differential equations makes use of insights on how to prove all invariants of differential equations [PT18, PT20]. Tactical implementations allow KeYmaera X to soundly reduce ODE invariance questions to a small number of core ODE axioms and real arithmetic. The proof tactic is optimized for fast proofs of commonly used invariants, e.g., barrier certificates [PJP07]. All real arithmetic questions that arise in the proofs are rigorously checked, *including* the ones that arise from the use of barrier certificates. This guarantees that any barrier certificate that proves with KeYmaera X is a *true* barrier certificate, rather than the result of numerical or floating-point errors.

To prove properties of differential equations, KeYmaera X combines an axiomatic differential equation solver [Pla17] and local fixedpoint computation for differential invariants [PC09a] with tactics based on differential equation axiomatization [PT18, PT20], and Pegasus: a toolbox for automatically generating continuous invariants for systems of ordinary differential equations. Given a system of ODEs subject to an evolution domain constraint, a set of initial states, and a set of unsafe states, Pegasus will attempt to automatically generate a continuous invariant that

[1] This dℒ uniform substitution calculus is also formally verified in Isabelle/HOL and Coq [BRV+17].

is sufficient to prove that the ODE cannot continuously evolve into an unsafe state from any of its initial states while respecting the evolution constraint. Pegasus [SMT+19] is implemented in Mathematica and is able to connect to Matlab; at present it implements an array of techniques from qualitative analysis and discrete abstraction [SGJP16] for constructing continuous invariants and is additionally capable of searching for *barrier certificates* (using both sum-of-squares optimization [PJ04] and linear programming [SC+13]), as well as *Darboux polynomials* (by making use of algorithms developed for the *Prelle-Singer* procedure [Man93]). The methods for continuous invariant generation within Pegasus are deployed in a targeted fashion, and depend on the nature of the verification problem: the important features of the problem are extracted by the tool in a pre-processing step which suggests promising strategies for invariant generation base on those features.

KeYmaera 3. KeYmaera 3 [PQ08] is the previous generation theorem prover for differential dynamic logic $\mathsf{d}\mathcal{L}$. Unlike its successor KeYmaera X, the older KeYmaera 3 directly implements a sequent calculus for differential dynamic logic [Pla08], instead of a uniform substitution calculus. What KeYmaera X implements from a few simple modular axioms, KeYmaera 3 uses several dedicated proof rules for [Pla08, Pla10a, Pla12b]. This leads to a more directly usable but substantially bigger soundness-critical prover kernel of about 66000 lines of code written in a mix of Java and Scala. In some cases, one single proof rule use, e.g., for solving differential equations in KeYmaera 3 corresponds to thousands of axiom uses in KeYmaera X. The impact on soundness, however, is that the ODE solver of KeYmaera 3 is trusted while that of KeYmaera X is not trusted, because each of its outputs is verified with a proof.

For proof automation, KeYmaera 3 implements a simple but fast fixpoint loop [PC09a] for generating loop invariants of hybrid systems and differential invariants of differential equations. It provides an array of different SMT strategies for splitting real arithmetic subquestions [Pla10b]. Changing proof search procedures in KeYmaera 3 (beyond choosing from the list of predefined ones) is significantly more complicated and, notably, soundness-critical.

HHL Prover. HHL prover [WZZ15] is an interactive theorem prover for verifying hybrid systems modelled by Hybrid CSP (HCSP) [He94, ZWR96]. HCSP is an extension of CSP by introducing differential equations for modeling continuous evolutions and interrupts for modeling interaction between continuous and discrete dynamics. HHL prover implements the Hybrid Hoare Logic (HHL) [LLQ+10], a Hoare style specification logic for HCSP, in the proof assistant Isabelle/HOL. However, as the HHL defined in [LLQ+10] is not compositional with respect to parallel composition, HHL prover can only handle restricted forms of parallel processes. *This version of the HHL prover is used for the nonlinear models, rollercoaster, and the lunar lander control program benchmarks.*

Recently, we implemented in HHL prover a compositional trace-based specification logic for HCSP, and used it to verify some examples involving combinations of ODE, interrupt, repetition, and parallel composition. Traces for both sequential and parallel HCSP processes consist of lists of trace blocks, and describe executions of a sequential or parallel process. For sequential processes, there are three types of trace blocks: τ-blocks for internal actions, input and output blocks for communication, and ODE blocks. For parallel processes, the trace blocks are τ-blocks for internal action on a single process, IO blocks for communication between two processes, and wait blocks that allow waiting or ODE evolution on each process. A list of sequential traces can be combined into a parallel trace, considering the synchronization of communication events, the interleaving of discrete actions occurring at the same time, and the conjunction of ODE trajectories.

A big-step semantics is defined for sequential processes, where $(c, tr) \Rightarrow tr'$ means executing process c with starting trace tr may terminate with the final trace tr'. Hoare triples (for partial correctness) is then defined as follows:

$$\{P\}\ c\ \{Q\} \Longleftrightarrow \forall tr\ tr'.\ P(tr) \longrightarrow (c, tr) \Rightarrow tr' \longrightarrow Q(tr')$$

We defined inference rules for Hoare triples based on traces, which enables reasoning about HCSP processes in a compositional way. *The new system is tested on a benchmark consisting of small HCSP programs involving combinations of ODE, interrupt, repetition, and parallel composition.*

Isabelle/HOL/Hybrid-Systems-VCs. These verification components for hybrid systems are part of Isabelle/HOL's Archive of Formal Proofs [HyM19, HyM20] (`https://www.isa-afp.org/entries/Hybrid_Systems_VCs.html`). Instead of deeply embedding a dynamic logic in the proof assistant, the components are based on a shallow embedding of Kleene algebras [GS16] that handle the program structure. Their instantiations to relational or state transformer semantics allow them to also model program assignments and differential equations [HyMS19], while the infrastructure to handle ODEs comes from a separate AFP-entry [IH12].

The approach has been designed to be flexible and versatile. When combined with Kleene algebras with tests (KAT), the components generate a Hoare-Logic and a Morgan-style refinement calculus for hybrid programs [FyMS20]. Alternatively, using modal Kleene algebras (MKA) provides predicate transformer semantics with weakest liberal preconditions for hybrid programs. In both settings, the flow of the differential equations, if known, can either be directly written in the specification or be certified with respect to existence and uniqueness. If the flow is unknown, there is also infrastructure to reason indirectly with invariant sets. Moreover, by formalising and proving soundness of the rules of dL, the components can reason in the style of this logic [HyMS19]. Finally, the components work on an open platform whose extension can allow them to reason directly with expressions beyond the scope of traditional dL. For instance, by implementing matrices in the proof assistant, the components can encode linear systems and their general solutions as operations between matrices and vectors [HyM20].

One consequence of this openness and generality is that the automation of the components highly depends on the solvers and decision procedures available in Isabelle/HOL. Yet, this also means that many improvements to the automation in the proof assistant will reverberate in the components. Another consequence is that Isabelle/HOL has to certify invariants and solutions to ODEs, but this increases trust in the verification procedure.

4 Benchmarks

One of the strengths of hybrid systems theorem proving as a verification technique is its support for combined automated and interactive verification steps as well as its applicability to proof search and proof checking. The benchmark examples were analyzed in three modes:

Automated The specification is the only input to the theorem prover. Proofs and counterexamples are obtained fully automated to highlight the capabilities of theorem provers in terms of invariant generation, proof search, and proof checking.

Hints Known design properties of the system, such as loop invariants and invariants of differential equations, are annotated in the model and allowed to be exploited during an otherwise fully automated proof to highlight the capabilities of theorem provers in terms of proof search and proof checking.

Scripted User guidance with proof scripts is allowed to highlight the capabilities of theorem provers in terms of proof checking.

The benchmark examples are structured into 5 categories: hybrid systems design shape examples to test for system design variations at a small scale, nonlinear continuous models to test for continuous invariant construction and proving capabilities, hybrid game examples to test adversarial dynamics, hybrid systems case studies to test for prover scalability, and counterexamples to test for capabilities in disproving wrong statements.

Experimental setup. KeYmaera X 4.6.3 and KeYmaera X 4.8.0 (in automated (A), hints (H), and scripted (S) mode) and KeYmaera 3 (in automated (A) mode) participated on all benchmark sets and were executed on the same machine (2013 Mac Pro with 6-core Intel Xeon E5 3.5GHz and 28GB memory; KeYmaera X uses a single core), and therefore their computation times are directly comparable. The Isabelle/HOL/Hybrid-Systems-VCs participated in scripted (S) mode only in the hybrid systems design shapes category and in the European train control system case study benchmark (two-core 2.5 GHz Intel Core i5, 8 GB 1600 MHz DDR3). HHL Prover participated with the Chinese train control system, lunar lander descent guidance, and roller-coaster safety case studies, as well as on a subset of the hybrid systems design shapes and the nonlinear continuous models. The execution time measurements were taken separately on a fresh prover instance for each example in the benchmark set. Proof attempts were aborted after a category-specific timeout, well above the longest successful solution in the category. The competition results are presented with *accumulated execution times* after examples are ranked according to their execution time.

4.1 Hybrid Systems Design Shapes

Category overview. In this category (unmodified from 2018 [MST+18] and 2019 [MST+19]), basic examples[2] test for proof automation techniques for a large variety of system designs: event-triggered systems, time-triggered systems, systems with nested loops and differential equations, and systems with model-predictive control. Instead of focusing on particularly complex systems, this set of examples strives at a certain degree of coverage of qualitatively different kinds of systems and their different typical shapes. The benchmark examples are grouped as follows:

Static semantics correctness 9 examples with various sequential orders and nested structures of assignments, differential equations, and loops.

Dynamics 30 examples with differential equations ranging from solvable to nonlinear.

LICS Tutorial 9 dℒ tutorial examples [Pla12a] ranging from basic time-triggered motion control to model-predictive control.

STTT Tutorial 12 dℒ modeling tutorial examples [QML+16] ranging from basic discrete event-triggered and time-triggered control for straight-line motion to speed control with a trajectory generator and lane-keeping with two-dimensional curved motion.

Competition results. The participants in the Hybrid Systems Design Shapes category include the KeYmaera X family of provers and the Isabelle/HOL verification components.

KeYmaera X family. In the KeYmaera X family, proof attempts were aborted after a timeout of 300 s in the basic category, with the longest successful solution after about 18 s

[2] https://github.com/LS-Lab/KeYmaeraX-projects/blob/master/benchmarks/basic.kyx

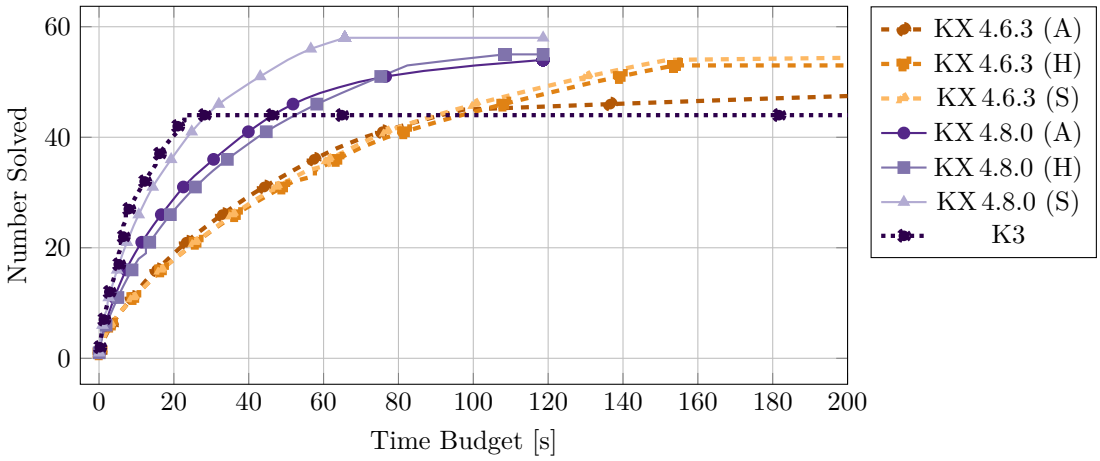

Figure 1: Computation times in the KeYmaera family of provers: Basic benchmark examples. Ranked accumulated time budgets [s], which are the number of examples solved within a total accumulated time budget; truncated at 200 s for plot separation.

(KeYmaera X 4.8.0) and 245 s (KeYmaera X 4.6.3). The results for the basic category in terms of accumulated execution times are shown in Fig. 1.

KeYmaera X 4.8.0 vastly improved performance: automation is now on par with prior scripted proofs, while proof checking duration was considerably lowered with an improved and restructured prover kernel. As an interesting side note, Fig. 1 shows that not all proof hints available in the input models present the most effective way to prove those models, since in KeYmaera X 4.8.0 full automation without proof hints finds some faster proofs than proofs from hints.

Isabelle/HOL verification components. The Isabelle/HOL verification components successfully proved 52 of the 60 examples without trusting external tools. Otherwise, the number of verified problems grows to 55. On commodity hardware (CPU: two-core 2.5 GHz Intel Core i5, memory: 8 GB 1600 MHz DDR3), Isabelle takes about 5 min to load the verification components, and once loaded, it takes on average 0.8625 s to check each of the 55 solved benchmark problems. For many examples, their verification required extra lemmas to help the proof assistant discharge all obligations. In most cases, we certified these extra requirements in Isabelle and used them in the corresponding verification proofs. Yet, the 8 not-fully verified problems needed lemmas that we could not prove during the time-frame of the competition. Although for 3 of those 8, the missing properties were facts of real-number arithmetic. Thus, we guaranteed these facts in Mathematica 12.1 and by asserting them in Isabelle, the components proved the corresponding verification problems. Three of the remaining problems needed major theorems from analysis like the fact that differentiable functions are Lipschitz-continuous. Also, we suspect that a different formalisation of Picard-Lindelöf's theorem is needed for their verification. The other two examples require addendums to handle formulas of $\mathsf{d}\mathcal{L}$ different from the standard $\phi \rightarrow [\alpha]\psi$.

4.2 Nonlinear Continuous Models

Category overview. This set of 141 nonlinear continuous safety verification problems[3] is based on the problems proposed in [SGJ16] and significantly extended from [MST+18, MST+19]. The problems in this benchmark set were gathered from published papers in the area of continuous safety verification and invariant generation for nonlinear systems ([DGXZ17, LZZ11, DCKB17, SGS14, SGJP16]). The bulk of the problems in the benchmark set feature planar (i.e., 2-dimensional) polynomial systems of ODEs in which the safety property is known to hold for unbounded time. The ODEs are furthermore autonomous (i.e., do not explicitly depend on the independent time variable t); this fact presents no real restriction since non-autonomous ODEs can be brought into autonomous form by augmenting the dynamics with $t' = 1$. Certain non-polynomial systems of ODEs can likewise be brought into polynomial form by introducing fresh variables in a process called *re-casting* [SV87]. While we stress that the existing set of nonlinear polynomial ODE safety benchmarks can in no way be said to be representative (owing to its small size), the general class of problems which fits into this category is highly important.

Example 4.1. The nonlinear system from [DLA06, Ex. 5.2. ii] that was shown in (2) has the following dynamics:

$$x' = 2x - 2xy,$$
$$y' = 2y - x^2 + y^2.$$

Taking the initial states to be $-\frac{4}{5} < x < -\frac{1}{3} \wedge -1 \leq y < 0$ and $(x = 0 \wedge y = 0) \vee x + y > 1$ to be the forbidden states, the verification problem is illustrated in Fig. 2.

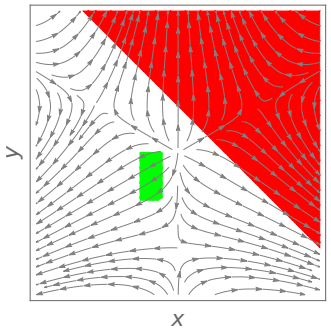

Figure 2: Nonlinear continuous safety verification problem. No initial state (green rectangle) can evolve into unsafe states (red half-plane) along the trajectories.

Competition results. The participants in the Hybrid Systems Case Study category include the KeYmaera family of provers. Proof attempts in the nonlinear category were aborted after a timeout of 300 s, above the longest successful solution of about 98 s (KeYmaera X 4.8.0) and 208 s (KeYmaera X 4.6.3).

Fig. 3 plots the accumulated execution times for the nonlinear category after examples are ranked according to their execution time. KeYmaera X 4.8.0 vastly improves performance, albeit at the expense of not proving a small number of previously solved examples with proof scripts and hints. More importantly, full automation improved in terms of reduced computation time as well as in the number of solved examples. A major next step in invariant generation in the nonlinear category is expected to exploit finite-horizon methods and thus additionally certify their results with proofs.

4.3 Hybrid Games

Category overview. This set of 3 simple examples[4] with adversarial dynamics tests theorem prover applicability to two-player game examples in differential game logic [Pla15]. Unlike in dℒ (where all non-determinism is either fully adversarially in safety proofs, or fully cooperative in

[3] https://github.com/LS-Lab/KeYmaeraX-projects/blob/master/benchmarks/nonlinear.kyx
[4] https://github.com/LS-Lab/KeYmaeraX-projects/blob/master/benchmarks/games.kyx

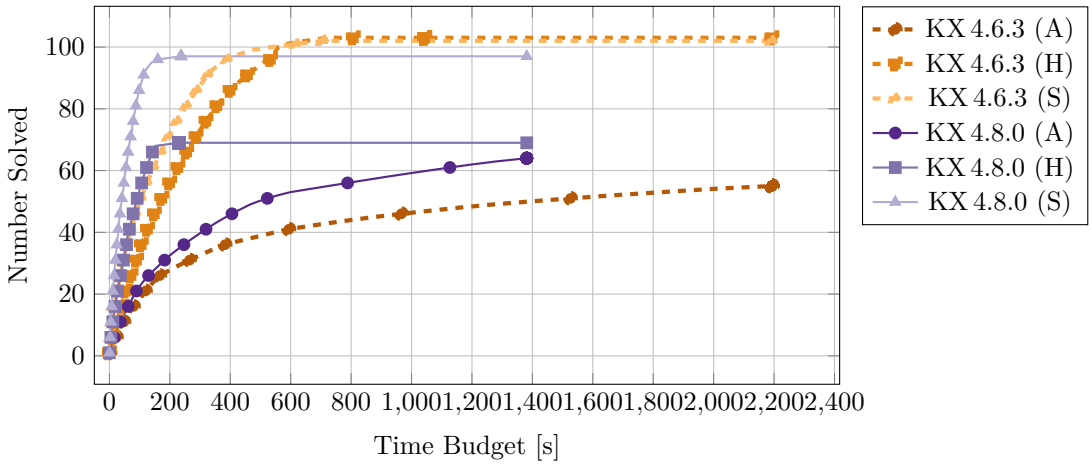

Figure 3: Computation times in the KeYmaera family of provers: Nonlinear benchmark examples. Ranked accumulated time budgets [s], which are the number of examples solved within a total accumulated time budget

liveness proofs), in differential game logic the responsibility for resolving non-determinism is attributed to players (angel and demon). For example, in the following formula, non-deterministic repetition is resolved by demon as indicated by $\alpha^\times$ (instead of α^*), while the non-deterministic choice remains angel's responsibility.

$$x = 0 \rightarrow \langle((x := 0 \cup x := 1))^\times\rangle x = 0$$

With $\langle\cdot\rangle$, we indicate that we want to know if angel has a winning strategy, with $[\cdot]$ whether demon has a winning strategy.

Responsibility for choices can be switched between players with a *game duality* operator α^d. Operator α^d can be thought of as turning the game board, so that all choices previously being angel's become demon's choices and vice versa. Demonic operators $\cap, \times$ are expressed by duality from their angelic counterparts $\cup, *$ as follows:

$$\alpha^\times \equiv \left((\alpha^d)^*\right)^d$$
$$\alpha \cap \beta \equiv (\alpha^d \cup \beta^d)^d$$

In ASCII syntax, choices are either expressed using UTF-8 symbols $\cap, \times$ or with operator α^d rendered in ASCII as `^@`, so the above example reads as follows:

```
x=0 -> <{{{x:=0;++x:=1;}^@}*}^@> x=0
```

Competition results. The participants in the Hybrid Games category include KeYmaera X 4.6.3 and KeYmaera X 4.8.0. Owing to the simple nature of the games examples, KeYmaera X 4.8.0 solved all three examples in less than 4 s in scripted mode. The hybrid systems proof automation of KeYmaera X 4.8.0 is capable of solving two of the examples, but more games automation is needed to truly search for winning strategies in games.

4.4 Hybrid Systems Case Study Benchmarks

Category overview. The benchmark examples in this category are selected to test theorem provers for scalability and efficiency on examples of a significant size and interest in applications and remained unchanged from [MST+19]. The benchmark examples[5] are inspired from prior case studies on train control [PQ09, ZLW+13], flight collision avoidance [PC09b], robot collision avoidance [MGVP17], a lunar lander descent guidance protocol [ZYZ+14], and rollercoaster safety [BLCP18].

European train control system (ETCS). This benchmark on automated train control bases on the safety analysis [PQ09] of the cooperation protocol in the European Train Control System [ERT02, DHO03], which specifies the interaction between an automated train protection system and a radio-block controller. The radio-block controller (purely discrete dynamics) may at any time issue speed limits that take effect at certain positions; the train must respect these speed limits (hybrid dynamics of train controller and train motion).

E-1 (ETCS: Essentials) Describes the core safety theorem: a time-triggered train controller never violates the posted speed limit.

E-2 (ETCS: Proposition 1 (Controllability)) Describes the motion of a train on brakes and translates it into a stopping distance. Tests a prover's ability to show equivalence between a hybrid systems specification in dℒ and it's core information in terms of stopping distance in real arithmetic.

E-3 (ETCS: Proposition 4 (Reactivity)) Describes the motion of a train when accelerating for a bounded amount of time and the necessary distance to a full stop. Tests a prover's ability to work with universally quantified assumptions and/or analyze programs in the context of universally quantified input.

The benchmark tests a prover's ability to handle dℒ safety properties (modal formulas) in various places of a specification, for example, as proof obligations and as assumptions.

Chinese train control systems (CTCS). This case study is about modeling and verification of a combined operational scenario of Chinese Train Control System Level-3 (CTCS-3). It originates from an under-specification error of the System Requirements Specification (SRS) of CTCS-3, revealed during a spot testing of the system, which caused a train to stop unexpectedly. It has been studied in [ZLW+13, ZZW+13, ZZWF15] and the failure was reproduced by simulation and also formally verified.

The combined scenario integrates the movement authority (MA) scenario, the level transition (from CTCS-2 to CTCS-3) scenario, as well as the mode transition (from Full Supervision mode to Calling On mode, FS to CO for short) scenario of CTCS-3. The combined scenario is shown in Fig. 4, which occurs under the following situation:

- The train has got enough MA to complete the combined scenario, and
- There are two adjacent segments in the MA, divided by location x_2. At x_2, the level transition from CTCS-2 to CTCS-3, and the mode transition from FS to CO, will occur simultaneously, and
- The train starts to move at location ST, and has an agreement from RBC (Radio Block Center) to start level transition at x_1 and complete the level transition at x_2.

According to the SRS, the combined scenario is required to satisfy a *liveness property*: the train can eventually move beyond the location x_2 with a positive speed, with both the level transition and mode transition completed successfully.

[5] https://github.com/LS-Lab/KeYmaeraX-projects/blob/master/benchmarks/advanced.kyx

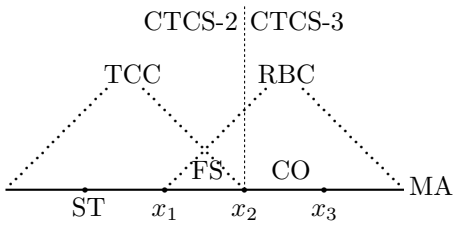

Figure 4: A combined scenario of CTCS-3.

However, the under-specified SRS fails to guarantee the liveness property. Basically, for safety reasons, to switch from FS mode to CO mode under CTCS-3, the driver's confirmation is required before the switching point x_2 to upgrade the speed limit of the CO mode, which is originally set to 0. However, in the old version of the SRS, such a confirmation request is not explicitly specified to be issued to the driver during a region where the train is co-supervised by both CTCS-2 and CTCS-3 (x_1 to x_2 in Fig. 4). As a result, the speed limit of the CO segment cannot be upgraded and remains 0, which forces the train to stop at x_2. Thus the verification objective for this case study is to prove on the underspecified model the *negation* of the liveness property, that is, the train must stop at x_2.

Roundabout air traffic conflict resolution (ATC). Air traffic conflict resolution maneuvers with curved flight dynamics exhibit nontrivial interactions of discrete and continuous dynamics. The roundabout benchmark [PC09a] is based on [TPL+96, TPS98, HHMW00, MF01, DPR05, PC09b, PKV09] to analyze collision freedom of planar roundabout maneuvers in air traffic control that should guarantee safe spatial separation of aircraft throughout their flight. The scale of this benchmark can be adjusted easily with the number of aircraft involved in the conflict resolution maneuver: additional aircraft increase the number of variables in the benchmark and introduce additional invariants that must be found, but analysis is separable into pairwise collision freedom questions.

A-2 (ATC: 2 Aircraft Tangential Roundabout Maneuver) Describes the circular conflict resolution of two aircraft in a planar roundabout collision avoidance maneuver.

A-3 (ATC: 3 Aircraft Tangential Roundabout Maneuver) Circular conflict resolution of three aircraft in planar roundabout collision avoidance maneuvers. Safety of the entire system is collision-freedom between all three aircraft pairs.

A-4 (ATC: 4 Aircraft Tangential Roundabout Maneuver) Circular conflict resolution of four aircraft in planar roundabout collision avoidance maneuvers. Safety of the entire system is collision-freedom between all six aircraft pairs.

The benchmark tests a prover's ability to analyze nested loops and multiple nonlinear differential equations. At larger numbers of aircraft it also tests the scale of reasoning about nonlinear dynamics by identifying and splitting analysis into isolated sub-questions.

Robot collision avoidance (RX). This benchmark bases on [MGVP17] and analyzes obstacle avoidance in ground robot navigation. The benchmark uses models and safety properties to analyze collision avoidance safety in the presence of stationary obstacles and moving obstacles.

The resulting real arithmetic formulas describing the Euclidian distance between robot and obstacle after symbolic execution are challenging for current solvers and may require overapproximation and simplification in the theorem prover steering the backend decision procedures.

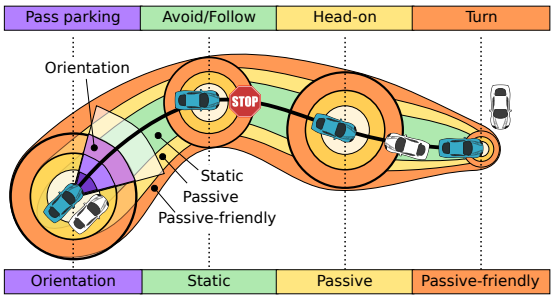

Figure 5: Robot collision avoidance properties: benchmark tests static safety and passive safety.

R-1 (Robot collision avoidance: static safety) ensures that no collisions can happen with stationary obstacles. Tests a prover's ability to handle mixed solvable (longitudinal robot acceleration) and nonlinear (rotational robot motion) continuous dynamics, and its ability to overapproximate norms (Euclidian distance overapproximated to infinity norm).

R-2 (Robot collision avoidance: passive safety) ensures that no collisions can happen with stationary or moving obstacles while the robot moves. The size of the resulting real arithmetic formulas are challenging for current solvers even after overapproximation of Euclidian distances. Tests a prover's ability to steer backend decision procedures by selecting relevant assumptions, using monotonicity arguments to eliminate variables, and simplify arithmetic.

This benchmark tests a prover's ability to analyze mixed solvable and nonlinear differential equations, overapproximation of norms, and arithmetic simplifications.

Lunar Lander Descent Guidance (LLDG). The lunar lander control program is a closed loop system, which is composed of the lander's dynamics and the guidance program for the slow descent phase. The guidance program is executed periodically with a fixed sampling period. At each sampling point, the current state of the lander is measured by inertial measurement unit or various sensors. Processed measurements are then input into the guidance program, which outputs control commands, e.g. the magnitude and direction of thrust, to be imposed on the lander's dynamics in the following sampling cycle. The mathematical description of the lander's dynamics as well as the guidance program of the slow descent phase can be found in [ZYZ+14, ZWZ16].

Rollercoaster Safety (RCS). The rollercoaster safety case study [BLCP18] is a benchmark in component-based verification combining smaller-scale components with non-trivial continuous dynamics to a full large-scale hybrid system. The components represent motion of a coaster car along geometrical primitives (straights, arcs) that can be connected to form complicated track shapes of varying scale.

Competition results. The participants in the Hybrid Systems Case Study category include the KeYmaera family of provers, Isabelle/HOL verification components, and HHL prover.

KeYmaera theorem provers Proof attempts in the hybrid systems case study category were aborted after a timeout of 1500 s, with the longest successful proof after about 340 s (KeY-

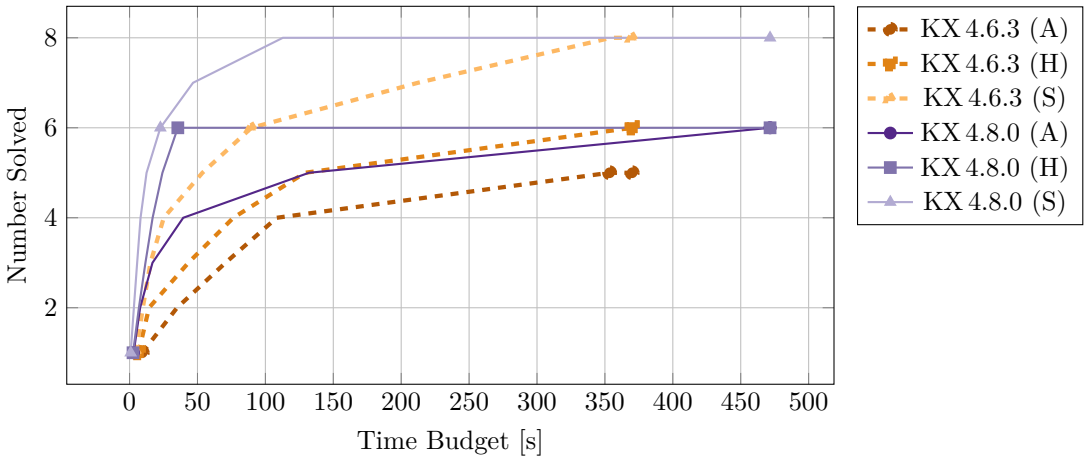

Figure 6: Computation times in the KeYmaera family of provers: Case study benchmark examples. Ranked accumulated time budgets [s], which are the number of examples solved within a total accumulated time budget

maera X 4.8.0) and 245 s (KeYmaera X 4.6.3). The proof durations in the KeYmaera family of provers is summarized in Fig. 6. The trend seen in hybrid systems design shapes and nonlinear examples continues: KeYmaera X 4.8.0 considerably reduced computation time with full automation at the level of proving from hints in KeYmaera X 4.6.3. Handling real arithmetic, however, remains a major challenge in large case studies, where at present human insight via proof scripts makes a considerable difference in provability and proof duration. Completeness improvements in the tactics of KeYmaera X 4.8.0, which helped increase automation in other categories, now retain arithmetic facts that were unintentionally discarded in KeYmaera X 4.6.3: discarding unnecessary facts helps the backend arithmetic tools to handle real arithmetic questions faster, which explains why proof duration increased in some case studies. Additional automation to mimic human reasoning is required to scale up hybrid systems proving further.

Isabelle/HOL verification components The Isabelle/HOL verification components successfully verified without external aid the ETCS: Essential benchmark problem. The components also verified the ETCS Controllability problem but by asserting (without certifying) a real-arithmetic fact. Finally, we did not manage to write a direct translation of the reactivity part of the ETCS case study from KeYmaera syntax to our components within the time-frame of the competition.

HHL Prover The CTCS case study is identical to the previous edition [MST+19]: a Stateflow/Simulink model has been built for the combined scenario in the CTCS-3 case study. Applying the tool Sim2HCSP to the Simulink/Stateflow model, seven files were generated which describe the HCSP model as well as the goal to be verified. Then using HHL Prover, the goal was proved successfully as a theorem, taking 59 seconds to finish on the M_{hhl} platform with Intel Core i7-4790 CPU 3.60GHZ and 16GB memory. In particular, during the interactive proof process, certain differential invariants were manually fed into the HHL specification.

In the LLDG case study, the entire Isabelle theory including the model, specification, and proof for the entire example is 327 lines long. By applying HHL prover, the unproven subgoals

related to differential invariants are transformed to a set of SOS constraints with respect to the user-defined invariant template, and then the SOS-based invariant generator is invoked on these constraints to synthesize a satisfying invariant.

In the RCS case study, the rollercoaster example [BLCP18] was converted to HCSP. The conversion is natural, as the differences between $\mathsf{d}\mathcal{L}$ and HCSP do not produce any problems. The proof makes use of invariant checking using Redlog, as well as the newly added differential ghost rule. The entire Isabelle theory (including the model, specification, and proof for all ten parts of the example) is 1141 lines long.

For the trace-based logic, some examples of sequential and parallel processes involving combinations of ODE, interrupt, repetition, and parallel composition are verified in Isabelle. In case of processes containing repetition, the invariant is an inductively defined assertion on traces, describing the set of traces that can occur after any finite number of repetitions. Defining and reasoning about such inductively defined assertions is one of the main challenges of the verification. Prover for trace-based logic also builds upon the existing analysis and ODE libraries in Isabelle to provide foundational proofs about properties of ODE evolution.

5 Conclusion and Outlook

The hybrid systems theorem proving friendly competition focuses on the characteristic features of hybrid systems theorem proving: flexibility of programming language principles for hybrid systems, unambiguous program semantics, and mathematically rigorous logical reasoning principles.

The benchmark examples are extended over previous years, especially in the nonlinear category to help scale hybrid systems theorem proving automation to more complicated dynamics, and with an entirely new games category to account for adversial dynamics. The hybrid systems theorem proving category allows tools to choose their operating mode on the spectrum from fast proof checking of scripted proofs, hint-supported proof search and checking, to full automation. The results in the KeYmaera X family of provers are summarized in Fig. 7, Fig. 8, and Fig. 9.

The progress over previous years is encouraging, but handling real arithmetic remains a major challenge in proof automation, since hybrid systems proofs not only depend on real arithmetic facts when closing proofs, but also when searching for loop invariants and invariant conditions of differential equations.

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We thank the entire Logical Systems Lab at Carnegie Mellon University for their many contributions and suggestions to KeYmaera X and its associated tools.

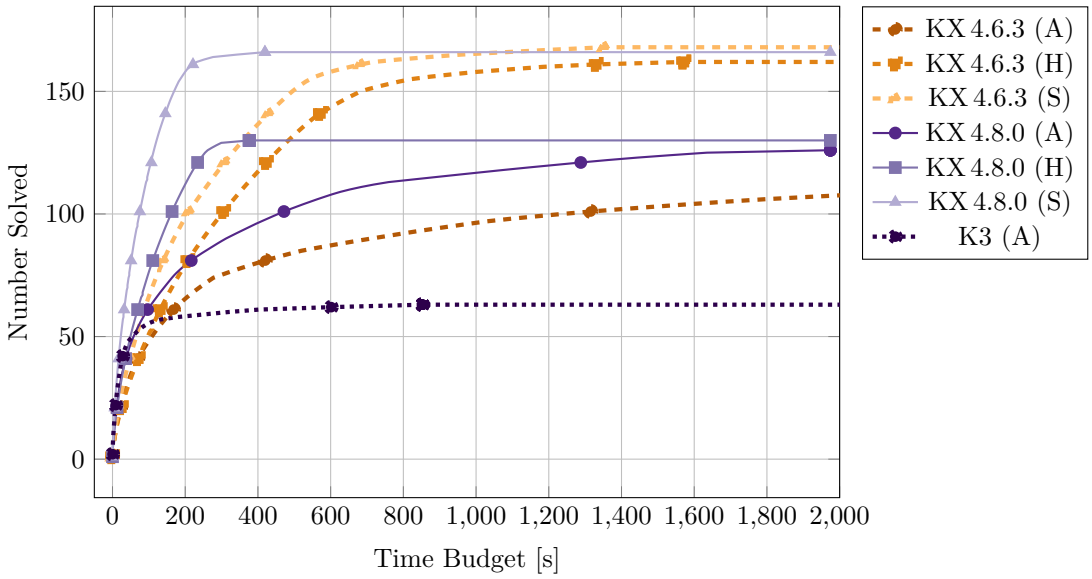


Figure 7: Ranked accumulated time budgets [s]: number of examples solved in total accumulated time budget (steeper is better). Result summary: KeYmaera X 4.8.0 closed the performance gap to KeYmaera 3 (curves are steeper longer). KeYmaera 3 solves less examples, especially among those with nonlinear dynamics (this effect is more pronounced than in [MST+19], since the number of nonlinear examples in the benchmark set increased further). KeYmaera X scales better; hints and scripts increase the number of solved examples and reduce computation time.

References

[BLCP18] Brandon Bohrer, Adriel Luo, Xue An Chuang, and André Platzer. CoasterX: A case study in component-driven hybrid systems proof automation. *IFAC-PapersOnLine*, 2018. Analysis and Design of Hybrid Systems ADHS.

[BRV+17] Brandon Bohrer, Vincent Rahli, Ivana Vukotic, Marcus Völp, and André Platzer. Formally verified differential dynamic logic. In Yves Bertot and Viktor Vafeiadis, editors, *Certified Programs and Proofs - 6th ACM SIGPLAN Conference, CPP 2017, Paris, France, January 16-17, 2017*, pages 208–221, New York, 2017. ACM.

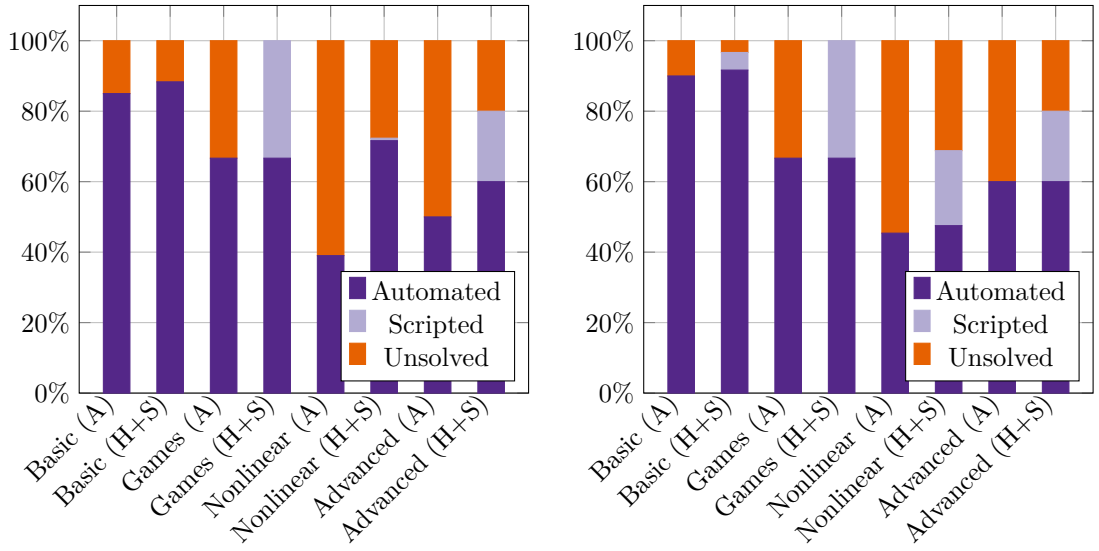
[DCKB17] Adel Djaballah, Alexandre Chapoutot, Michel Kieffer, and Olivier Bouissou. Construction of parametric barrier functions for dynamical systems using interval analysis. *Automatica*, 78:287–296, 2017.

[DGXZ17] Liyun Dai, Ting Gan, Bican Xia, and Naijun Zhan. Barrier certificates revisited. *J. Symb. Comput.*, 80:62–86, May 2017.

[DHO03] Werner Damm, Hardi Hungar, and Ernst-Rüdiger Olderog. On the verification of cooperating traffic agents. In Frank S. de Boer, Marcello M. Bonsangue, Susanne Graf, and Willem P. de Roever, editors, *FMCO*, volume 3188 of *LNCS*, pages 77–110. Springer, 2003.

[DLA06] Freddy Dumortier, Jaume Llibre, and Joan C Artés. *Qualitative Theory of Planar Differential Systems*. Springer, 2006.

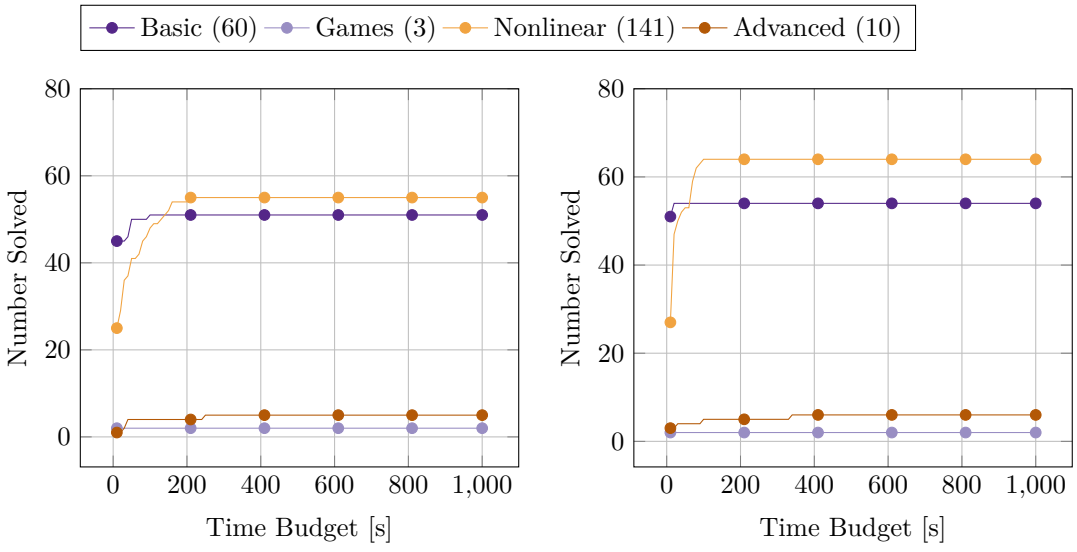
[DPR05] Werner Damm, Guilherme Pinto, and Stefan Ratschan. Guaranteed termination in the verification of LTL properties of non-linear robust discrete time hybrid systems. In Doron A. Peled and Yih-Kuen Tsay, editors, *ATVA*, volume 3707 of *LNCS*, pages 99–113. Springer,



(a) KeYmaera X 4.6.3: Number solved automated (A), hints (H), and scripted (S)

(b) KeYmaera X 4.8.0: Number solved automated (A), hints (H), and scripted (S)

Figure 8: Result summary: The trend started in [MST+19] continues: KeYmaera X 4.8.0 automation is now at the level of KeYmaera X 4.6.3 scripting; additional scripting in KeYmaera X 4.8.0 increases the number of solvable examples.



(a) KeYmaera X 4.6.3

(b) KeYmaera X 4.8.0

Figure 9: Result summary: Number of examples solvable fully automatically (A) with individual time budgets.

2005.

[ERT02] ERTMS User Group. UNISIG: ERTMS/ETCS system requirements specification. http://www.era.europa.eu, 2002. Version 2.2.2.

[FMBP17] Nathan Fulton, Stefan Mitsch, Brandon Bohrer, and André Platzer. Bellerophon: Tactical theorem proving for hybrid systems. In Mauricio Ayala-Rincón and César A. Muñoz, editors, *ITP*, volume 10499 of *LNCS*, pages 207–224. Springer, 2017.

[FMQ+15] Nathan Fulton, Stefan Mitsch, Jan-David Quesel, Marcus Völp, and André Platzer. KeYmaera X: An axiomatic tactical theorem prover for hybrid systems. In Amy Felty and Aart Middeldorp, editors, *CADE*, volume 9195 of *LNCS*, pages 527–538, Berlin, 2015. Springer.

[FyMS20] Simon Foster, Jonathan Julián Huerta y Munive, and Georg Struth. Differential hoare logics and refinement calculi for hybrid systems with isabelle/hol. In *RAMiCS 2020[postponed]*, pages 169–186, 2020.

[GS16] Victor B. F. Gomes and Georg Struth. Modal Kleene algebra applied to program correctness. In *FM 2016*, volume 9995 of *LNCS*, pages 310–325, 2016.

[He94] J. He. From CSP to hybrid systems. In *A Classical Mind, Essays in Honour of C.A.R. Hoare*, pages 171–189. Prentice Hall International (UK) Ltd., 1994.

[HHMW00] Thomas A. Henzinger, Benjamin Horowitz, Rupak Majumdar, and Howard Wong-Toi. Beyond HYTECH: hybrid systems analysis using interval numerical methods. In Nancy A. Lynch and Bruce H. Krogh, editors, *HSCC*, volume 1790 of *LNCS*, pages 130–144. Springer, 2000.

[HyM19] Jonathan Julián Huerta y Munive. Verification components for hybrid systems. *Archive of Formal Proofs*, 2019.

[HyM20] Jonathan Julián Huerta y Munive. Matrices for odes. *Archive of Formal Proofs*, 2020.

[HyMS19] Jonathan Julián Huerta y Munive and G. Struth. Predicate transformer semantics for hybrid systems: Verification components for Isabelle/HOL. arXiv:1909.05618 [cs.LO], 2019.

[IH12] Fabian Immler and Johannes Hölzl. Ordinary differential equations. *Archive of Formal Proofs*, 2012.

[LLQ+10] J. Liu, J. Lv, Z. Quan, N. Zhan, H. Zhao, C. Zhou, and L. Zou. A calculus for hybrid CSP. In *APLAS 2010*, volume 6461 of *LNCS*, pages 1–15. Springer, 2010.

[LZZ11] Jiang Liu, Naijun Zhan, and Hengjun Zhao. Computing semi-algebraic invariants for polynomial dynamical systems. In Samarjit Chakraborty, Ahmed Jerraya, Sanjoy K. Baruah, and Sebastian Fischmeister, editors, *EMSOFT*, pages 97–106. ACM, 2011.

[Man93] Yiu-Kwong Man. Computing closed form solutions of first order odes using the prelle-singer procedure. *Journal of Symbolic Computation*, 16(5):423–443, 1993.

[MF01] Mieke Massink and Nicoletta De Francesco. Modelling free flight with collision avoidance. In *ICECCS*, pages 270–280. IEEE Computer Society, 2001.

[MGVP17] Stefan Mitsch, Khalil Ghorbal, David Vogelbacher, and André Platzer. Formal verification of obstacle avoidance and navigation of ground robots. *I. J. Robotics Res.*, 36(12):1312–1340, 2017.

[MST+18] Stefan Mitsch, Andrew Sogokon, Yong Kiam Tan, André Platzer, Hengjun Zhao, Xiangyu Jin, Shuling Wang, and Naijun Zhan. ARCH-COMP18 category report: Hybrid systems theorem proving. In Goran Frehse, Matthias Althoff, Sergiy Bogomolov, and Taylor T. Johnson, editors, *ARCH18. 5th International Workshop on Applied Verification of Continuous and Hybrid Systems, ARCH@ADHS 2018, Oxford, UK, July 13, 2018*, volume 54 of *EPiC Series in Computing*, pages 110–127. EasyChair, 2018.

[MST+19] Stefan Mitsch, Andrew Sogokon, Yong Kiam Tan, Xiangyu Jin, Bohua Zhan, Shuling Wang, and Naijun Zhan. ARCH-COMP19 category report: Hybrid systems theorem proving. In Goran Frehse and Matthias Althoff, editors, *ARCH19. 6th International Workshop*

on Applied Verification of Continuous and Hybrid Systemsi, part of CPS-IoT Week 2019, Montreal, QC, Canada, April 15, 2019, volume 61 of *EPiC Series in Computing*, pages 141–161. EasyChair, 2019.

[PC09a] André Platzer and Edmund M. Clarke. Computing differential invariants of hybrid systems as fixedpoints. *Form. Methods Syst. Des.*, 35(1):98–120, 2009. Special issue for selected papers from CAV'08.

[PC09b] André Platzer and Edmund M. Clarke. Formal verification of curved flight collision avoidance maneuvers: A case study. In Ana Cavalcanti and Dennis Dams, editors, *FM*, volume 5850 of *LNCS*, pages 547–562, Berlin, 2009. Springer.

[PJ04] Stephen Prajna and Ali Jadbabaie. Safety verification of hybrid systems using barrier certificates. In *International Workshop on Hybrid Systems: Computation and Control*, pages 477–492. Springer, 2004.

[PJP07] Stephen Prajna, Ali Jadbabaie, and George J. Pappas. A framework for worst-case and stochastic safety verification using barrier certificates. *IEEE T. Automat. Contr.*, 52(8):1415–1429, 2007.

[PKV09] Erion Plaku, Lydia E. Kavraki, and Moshe Y. Vardi. Hybrid systems: from verification to falsification by combining motion planning and discrete search. *Form. Methods Syst. Des.*, 34(2):157–182, 2009.

[Pla08] André Platzer. Differential dynamic logic for hybrid systems. *J. Autom. Reas.*, 41(2):143–189, 2008.

[Pla10a] André Platzer. Differential-algebraic dynamic logic for differential-algebraic programs. *J. Log. Comput.*, 20(1):309–352, 2010.

[Pla10b] André Platzer. *Logical Analysis of Hybrid Systems: Proving Theorems for Complex Dynamics*. Springer, Heidelberg, 2010.

[Pla12a] André Platzer. Logics of dynamical systems. In *LICS*, pages 13–24, Los Alamitos, 2012. IEEE.

[Pla12b] André Platzer. The structure of differential invariants and differential cut elimination. *Log. Meth. Comput. Sci.*, 8(4:16):1–38, 2012.

[Pla15] André Platzer. Differential game logic. *ACM Trans. Comput. Log.*, 17(1):1:1–1:51, 2015.

[Pla17] André Platzer. A complete uniform substitution calculus for differential dynamic logic. *J. Autom. Reas.*, 59(2):219–265, 2017.

[Pla18] André Platzer. *Logical Foundations of Cyber-Physical Systems*. Springer, Switzerland, 2018.

[Pla19] André Platzer. Uniform substitution at one fell swoop. In Pascal Fontaine, editor, *CADE*, volume 11716 of *LNCS*, pages 425–441. Springer, 2019.

[PQ08] André Platzer and Jan-David Quesel. KeYmaera: A hybrid theorem prover for hybrid systems. In Alessandro Armando, Peter Baumgartner, and Gilles Dowek, editors, *IJCAR*, volume 5195 of *LNCS*, pages 171–178, Berlin, 2008. Springer.

[PQ09] André Platzer and Jan-David Quesel. European Train Control System: A case study in formal verification. In Karin Breitman and Ana Cavalcanti, editors, *ICFEM*, volume 5885 of *LNCS*, pages 246–265, Berlin, 2009. Springer.

[PT18] André Platzer and Yong Kiam Tan. Differential equation axiomatization: The impressive power of differential ghosts. In Anuj Dawar and Erich Grädel, editors, *LICS*, New York, 2018. ACM.

[PT20] André Platzer and Yong Kiam Tan. Differential equation invariance axiomatization. *J. ACM*, 67(1):6:1–6:66, 2020.

[QML+16] Jan-David Quesel, Stefan Mitsch, Sarah Loos, Nikos Aréchiga, and André Platzer. How to model and prove hybrid systems with KeYmaera: A tutorial on safety. *STTT*, 18(1):67–91, 2016.

[SC+13] Sriram Sankaranarayanan, Xin Chen, et al. Lyapunov function synthesis using handelman representations. *IFAC Proceedings Volumes*, 46(23):576–581, 2013.

[SGJ16] Andrew Sogokon, Khalil Ghorbal, and Taylor T Johnson. Non-linear continuous systems for safety verification (benchmark proposal). In *ARCH@CPSWeek*, volume 43, pages 42–51. EasyChair, 2016.

[SGJP16] Andrew Sogokon, Khalil Ghorbal, Paul B. Jackson, and André Platzer. A method for invariant generation for polynomial continuous systems. In Barbara Jobstmann and K. Rustan M. Leino, editors, *VMCAI*, volume 9583 of *LNCS*, pages 268–288. Springer, 2016.

[SGS14] Mohamed Amin Ben Sassi, Antoine Girard, and Sriram Sankaranarayanan. Iterative computation of polyhedral invariants sets for polynomial dynamical systems. In *CDC*, pages 6348–6353. IEEE, 2014.

[SMT+19] Andrew Sogokon, Stefan Mitsch, Yong Kiam Tan, Katherine Cordwell, and André Platzer. Pegasus: A framework for sound continuous invariant generation. In Maurice H. ter Beek, Annabelle McIver, and José N. Oliveira, editors, *Formal Methods - The Next 30 Years - Third World Congress, FM 2019, Porto, Portugal, October 7-11, 2019, Proceedings*, volume 11800 of *LNCS*, pages 138–157. Springer, 2019.

[SV87] Michael A. Savageau and Eberhard O. Voit. Recasting nonlinear differential equations as S-systems: a canonical nonlinear form . *Mathematical Biosciences*, 87(1):83 – 115, 1987.

[TPL+96] Claire Tomlin, George J. Pappas, John Lygeros, Datta N. Godbole, and Shankar Sastry. Hybrid control models of next generarion air traffic management. In Panos J. Antsaklis, Wolf Kohn, Anil Nerode, and Shankar Sastry, editors, *Hybrid Systems IV*, volume 1273 of *LNCS*, pages 378–404. Springer, 1996.

[TPS98] Claire Tomlin, George J. Pappas, and Shankar Sastry. Conflict resolution for air traffic management: a study in multiagent hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):509–521, Apr 1998.

[WZZ15] S. Wang, N. Zhan, and L. Zou. An improved HHL prover: an interactive theorem prover for hybrid systems. In *ICFEM 2015*, volume 9407 of *LNCS*, pages 382–399. Springer, 2015.

[ZLW+13] Liang Zou, Jidong Lv, Shuling Wang, Naijun Zhan, Tao Tang, Lei Yuan, and Yu Liu. Verifying chinese train control system under a combined scenario by theorem proving. In Ernie Cohen and Andrey Rybalchenko, editors, *Verified Software: Theories, Tools, Experiments - 5th International Conf., VSTTE 2013, Menlo Park, CA, USA, May 17-19, 2013, Revised Selected Papers*, volume 8164 of *LNCS*, pages 262–280. Springer, 2013.

[ZWR96] Chaochen Zhou, Ji Wang, and Anders P. Ravn. A formal description of hybrid systems. In Rajeev Alur, Thomas A. Henzinger, and Eduardo D. Sontag, editors, *Hybrid Systems III*, volume 1066 of *LNCS*, pages 511–530. Springer, 1996.

[ZWZ16] Naijun Zhan, Shuling Wang, and Hengjun Zhao. *Formal Verification of Simulink/Stateflow Diagrams - A Deductive Approach*. Springer, 2016.

[ZYZ+14] Hengjun Zhao, Mengfei Yang, Naijun Zhan, Bin Gu, Liang Zou, and Yao Chen. Formal verification of a descent guidance control program of a lunar lander. In Cliff B. Jones, Pekka Pihlajasaari, and Jun Sun, editors, *FM 2014: Formal Methods - 19th International Symposium, Singapore, May 12-16, 2014. Proceedings*, volume 8442 of *LNCS*, pages 733–748. Springer, 2014.

[ZZW+13] Liang Zou, Naijun Zhan, Shuling Wang, Martin Fränzle, and Shengchao Qin. Verifying Simulink diagrams via a Hybrid Hoare Logic prover. In *Proceedings of the International Conference on Embedded Software, EMSOFT 2013, Montreal, QC, Canada, Sept. 29 - Oct. 4, 2013*, pages 9:1–9:10, 2013.

[ZZWF15] Liang Zou, Naijun Zhan, Shuling Wang, and Martin Fränzle. Formal verification of Simulink/Stateflow diagrams. In *Automated Technology for Verification and Analysis - 13th International Symposium, ATVA 2015, Shanghai, China, Oct. 12-15, 2015, Proceedings*, pages 464–481, 2015.