General Topos Semantics for Higher-Order Modal Logic

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Topos-theoretic semantics for modal logic usually considers structures induced by a surjective geometric morphism $f : \mathcal{F} \to \mathcal{E}$. f restricts to an injective (complete) distributive lattice homomorphism

$$\Delta_A : \mathrm{Sub}_{\mathcal{E}}(A) \longrightarrow \mathrm{Sub}_{\mathcal{F}}(f^*A),$$

for each A in $\mathcal{E}$ and natural in A w.r.t. to pullback. Each Δ_A has a right adjoint Γ_A that composes with Δ_A to an endofunctor $\Delta_A\Gamma_A$ on the Heyting algebra $\mathrm{Sub}_{\mathcal{F}}(f^*A)$ that satisfies the axioms for an S4 modality on $\mathrm{Sub}_{\mathcal{F}}(f^*A)$. [3, 8, 9, 11]

Equivalently, regarding $f_*\Omega_{\mathcal{F}}$ as an internal frame in $\mathcal{E}$, one has internal adjoints $i \dashv \tau$,

$$\tau : f_*\Omega_{\mathcal{F}} \leftrightarrows \Omega_{\mathcal{E}} : i,$$

where i is the unique frame map from the initial frame $\Omega_{\mathcal{E}}$, and τ is the classifying map of $f_*(\top)$, with $1 : \top \to \Omega_{\mathcal{F}}$ the subobject classifier in $\mathcal{F}$. [7, 11]

The typical example is a geometric morphism of the form

$$\mathbf{Sets}^{|\mathbf{K}|} \longrightarrow \mathbf{Sets}^{\mathbf{K}},$$

for a preorder $\mathbf{K}$, induced by the inclusion $|\mathbf{K}| \to \mathbf{K}$ of the underlying set into $\mathbf{K}$. This yields "Kripke sheaf" semantics for modal logic. Here $\mathbf{K}$ is a preordered set of worlds, and each functor $P : \mathbf{K} \to \mathbf{Sets}$ gives a domain $P(k)$, for each $k \in \mathbf{K}$, with suitable comparison maps $P(k) \to P(l)$, whenever $k \leq l$ in $\mathbf{K}$. In this way each functor P determines a "Kripke sheaf", a first-order Kripke frame with varying domains of individuals. [2, 4, 12] Further examples include sheaf structures from the geometric morphism $\mathbf{Sets}/X \to \mathrm{Sh}(X)$, for a topological space X. [1, 5, 6]

In this talk, we will provide a slightly more general algebraic framework of modal structures in an arbitrary topos $\mathcal{E}$ with respect to which it is possible to build models of a certain system of (intuitionistic) higher-oder modal logic. The "geometric" models from before arise as a special case of the latter.

The data for such structures consist first of all of a complete internal Heyting algebra H in $\mathcal{E}$. Since $\Omega_{\mathcal{E}}$ is the initial frame in $\mathcal{E}$, there exists a unique frame homomorphism $i : \Omega_{\mathcal{E}} \to H$ that has a right adjoint τ, namely the classifying map of the top element $\top : 1 \to H$ of H (cf. [11, 10]. In what follows we will consider only those Heyting algebras H in $\mathcal{E}$ such that i is monic.

H interprets the logical operations via its Heyting structure, just as in standard algebraic semantics. Moreover, universal and existential quantification is modelled by indexed meets and

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joins in H, respectively, since H is assumed to be complete. For any object I in $\mathcal{E}$, I-indexed meets are defined to be given by a right adjoint

$$\forall_I : H^I \to H$$

to the canonical map $\Delta_I : H \to H^I$ that arises in turn from the unique map $I \to 1$ under the functor $H^{(-)}$. I-indexed joins are defined as a left adjoint to Δ_I.

A model in H is then given by objects M for each basic type in the language, while complex types are interpreted as usual using the cartesian closed structure in $\mathcal{E}$. The propositional type Prop is interpreted by H. The interpretation of non-modal formulas is the standard one except for the treatment of equality which is new. Each formula $\varphi(x_1, \ldots, x_n)$ in the free variables $x_1, \ldots, x_n$ is recursively assigned a map $M_1 \times \cdots \times M_n \xrightarrow{[\![\varphi]\!]} H$, where M_i interprets the type of x_i. For instance, $\forall y \varphi(x, y)$ is the map $X \xrightarrow{\lambda_Y [\![\varphi]\!]} H^Y \xrightarrow{\forall_Y} H$, where $\lambda_Y [\![\varphi]\!]$ is the exponential transpose of $[\![\varphi]\!]$, and X, Y interpret the respective types. The map $i \circ \tau : H \to H$ interprets the modal operator which satisfies the S4 laws by virtue of properties of the adjunction.

More importantly, for each type in the language, equality on the corresponding object M is given by the map

$$M \times M \xrightarrow{\delta_M} \Omega_{\mathcal{E}} \xrightarrow{i} H,$$

where i is the unique frame map from before, and δ_M is the classifying map of the diagonal $\langle 1_M, 1_M \rangle$.

It can be shown that when $H = f_* \Omega_{\mathcal{F}}$, for a geometric morphism f, then this definition of model coincides with the standard one associated with f. In particular, for equality we have

Lemma. *For any geometric morphism $f : \mathcal{F} \to \mathcal{E}$, and any object M in $\mathcal{E}$,*

$$i \circ \delta_M = \overline{\delta_{f^* M}},$$

where $\overline{\delta_{f^* M}}$ is the transpose (along $f^* \dashv f_*$) of the classifying map of the diagonal on $f^* M$.

This yields the following result, based on an already known fact for the geometric case [13]:

Proposition. *The extensionality principles*

$$\forall_x (f(x) = g(x)) \vdash f = g,$$

$$(p \Leftrightarrow q) \vdash p = q.$$

fail to be valid in the algebraic semantics described above.

The reason is precisely the interpretation of equality. However, we have

Proposition. *The following statements are valid in the algebraic topos semantics:*

$$\Box \forall_x (f(x) = g(x)) \vdash f = g,$$

$$\Box (p \Leftrightarrow q) \vdash p = q.$$

As a result, the higher-order S4 system that results from replacing propositional and functional extensionality by their modalized versions, respectively, is sound w.r.t. algebraic models in $\mathcal{E}$.

We will outline in detail why the non-modal versions fail and why the boxed versions are valid in the semantics. It is precisely the map $\tau : H \to \Omega$ from before which does the relevant work to repair the failure of the standard extensionality principles. In a way, thus, the modal operator as treated here turns out to be precisely what needs to be added to soundly interpret intuitionistic higher-order logic in a complete Heyting algebra in $\mathcal{E}$.

We conjecture, moreover, that this yields a complete semantics w.r.t. toposes for the outlined system of higher order S4 modal logic with the standard extensionality principles replaced by the modalized ones displayed above.

We conclude the talk by observing that every algebraic model in $\mathcal{E}$ arises from a geometric morphism from a suitable topos $\mathcal{F}$, i.e. $H = f_*\Omega_{\mathcal{F}}$, for $\mathcal{F}$ the topos of internal sheaves on H and f the canonical geometric morphism into $\mathcal{E}$.

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