Three-Valued Paraconsistent Logics with Subclassical Negation and Their Extensions

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May 20, 2021

THREE-VALUED PARACONSISTENT LOGICS WITH SUBCLASSICAL NEGATION AND THEIR EXTENSIONS

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Abstract. We first prove that any [conjunctive/disjunctive/implicative] 3-valued paraconsistent logic with subclassical negation (3VPLSN) is defined by a unique {modulo isomorphism} [conjunctive/disjunctive/implicative] 3-valued matrix and provide effective algebraic criteria of any 3VPLSN's being *subclassical*|being *maximally* paraconsistent|having no (inferentially) consistent non-subclassical extension implying that any [conjunctive/disjunctive]|conjunctive/"both disjunctive and {non-}subclassical"/"refuting *Double Negation Law*"|"conjunctive/disjunctive subclassical" 3VPLSN "is subclassical if[f] its defining 3-valued matrix has a 2-valued submatrix"|"is {pre-}maximally paraconsistent"|"has a theorem but no consistent non-subclassical extension". Next, any disjunctive/implicative 3VPLSN has no proper consistent non-classical disjunctive/axiomatic extension, any classical extension being disjunctive/axiomatic and relatively axiomatized by the "*Resolution* rule"/"*Ex Contradictione Quodlibet* axiom". Further, we provide an effective algebraic criterion of a [subclassical] "3VPLSN with lattice conjunction and disjunction"'s having no proper [consistent non-classical] extension but that [non-]inconsistent one which is relatively axiomatized by the *Ex Contradictione Quodlibet* rule [and defined by the product of any defining 3-valued matrix and its 2-valued submatrix]. Finally, any disjunctive and conjunctive 3VPLSN with classically-valued connectives has an infinite increasing chain of finitary extensions.

1. Introduction

Appearance of any logic/calculus (satisfying a property P) inevitably raises the issue of studying connections between the former and other ones (especially, those [not] satisfying P). Perhaps, a most representative connection is the extension one, especially because finding the lattice of extensions of a given logic/calculus is normally a technically quite non-trivial mathematical problem. This, in its turn, subsumes, at least, two logically quite significant particular instances. The first one is the question whether a logic/calculus can not be enhanced {at least, in no more than one way} (by extending with new — viz., non-derivable — rules [without premises]) but with retaining the property P, in which case it is said to be *[axiomatically] {pre}maximally|completely* P. The second one, equally dealing with the maximality point but from a more specific insight, concerns the issue of *structural completeness|completion*. (Recall that a logic/calculus is said to be *structurality complete*, provided any rule is derivable in it, whenever it is admissible in it, that is, adding the rule retains theorems — viz., derivable axioms. This means the maximality with respect to P, being the set of theorems of the logic/calculus, and so the factual deductive maximality|completeness of it, in this way becoming a most fundamental feature of it.) This property is [not] typical of [inferentially-consistent but not necessarily] classical logics (more precisely, two-valued classically-negative

2020 *Mathematics Subject Classification.* 03B20, 03B22, 03B50, 03B53, 03G10, 03F03.
Key words and phrases. logic; calculus; matrix; extension.

ones with a single distinguished value)[1] with[out] theorems. As for non-classical (in particular, many-valued) logics, the situation is far more ambiguous (even, for three-valued ones).

Within the framework of Paraconsistent Logic, P is paraconsistency — viz., refuting the *Ex Contradictione Quodlibet* rule. Then, maximal paraconsistency (versus it axiomatic version first observed in [22] for P^1) was first discovered in [13] for the *logic of paradox* LP [11] and then also proved in [16] for HZ [4] and for arbitrary expansions of Sugihara three-valued logic $\mathbb{S}_3$ [23] in [19]. And what is more, it has been proved for arbitrary conjunctive paraconsistent *subclassical* (viz., having a classical extension) three-valued logics (including all the *particular* logics mentioned above, and so providing a first proof of the maximal paraconsistency of P^1; cf. the reference [Pyn95 b] of [13]). The present study substantially enhances that one, at least, in the following essential respects. First of all, we provide an effective (in case of finitely many connectives)[2] algebraic criterion of the [pre]maximal paraconsistency of three-valued paraconsistent logics with subclassical negation (3VLPSN), among other things, positively inherited by their *three-valued* expansions (viz., enhancements by additional connectives), according to which any (three-valued expansion of any) conjunctive/"disjunctive {in particular, implicative}" 3VLPSN /"with[out] classical extensions" is [/pre]maximally paraconsistent. In this connection, note that, as opposed to presence of classical extensions, absence of these as well as both conjunctivity, disjunctivity, paraconsistency and subclassical negation are inherited by expansions. Therefore, as opposed to the reference [Pyn95 b] of [13], the present study is well-aaplicable to arbitrary three-valued expansions of all particular logics mentioned above. As an almost immediate observation, we also show that any 3VLPSN is *axiomatically* maximally paraconsistent that subsumes the axiomatic maximal paraconsistency of P^1 proved *ad hoc* in [22]. Likewise, we prove that any disjuctive/implicative 3VLPSN has a consistent disjuctive/axiomatic proper extension iff it has a classical extension, in which case this is the only one and is relatively axiomatized by the "*Resolution* rule"/ "*Ex Contradictione Quodlibet* axiom, being an implicative counterpart of the *Ex Contradictione Quodlibet* rule, that subsumes Theorem 6.3 of [12] equally subsuming [22]".

On the other hand, paraconsistency within the framework of Many-Valued Logic normally results from invoking a third truth value, say, $\frac{1}{2}$ (aside from the two classical ones: the [non-]distinguished $1[-1]$) to express inconsistency about assertions, in which case both the third value and its negation are set to be distinguished, while the classical negation typical of classical logic is retained on the classical values 0 and 1 — we naturally call such matrices *super-classical* here. It appears that this generic semantic schema (among other things, subsuming all the instances discussed above) exhausts all 3VLPSN, any super-classical matrix being uniquely determined by the logic defined by it, as we prove here, that well-justifies the present universal study.

Within the more general framework of Non-Classical Logic (including the narrower one of Subclassical Logic), P is presence of classical extensions. In this connection, we prove an equally effective algebraic criterion of 3VLPSN's satisfying it, according to which, in particular, any [conjunctive/disjunctive] 3VLPSN has at

[1]Properly speaking, within General Logic dealing with miscellaneous logics/calculi, Classical Propositional Logic/Calculus PC arises as rather the clone of functionally complete two-valued logics with a single distinguished value than any single specific logic. We equally follow this conventional paradigm here (even, without presuming the presence of functional completeness but merely of classical negation that naturally gives rise to the conception of *subclassical negation* adopted here.)

[2]From now on, such a reservation is presumed *tacitly/implicitly*, when leading the conversation toward effectiveness.

most one classical extension and if[f] its defining matrix has a two-valued classically-valued submatrix, that has proved especially valuable because of profound connections between maximal paraconsistency, presence/absense of theorems, [{non-}sub]classical [{(inferentially) consistent}] extensions and structural completeness of 3VLPSN discovered here. More precisely, according to an equally effective algebraic criterion of the structural completeness of [conjunctive] 3VLSPN, it implies [resp., is equivalent to] (aside from the quite obvious presence of theorems) absence of classical extensions [alone, for they are maximally paraconsistent and have theorems] as well as maximal paraconsistency. And what is more, presence of theorems of 3VLPSN with a classical extension is equivalent to absence of consistent non-subclassical extensions as well as implies maximal paraconsistency, and so holds for conjunctive/disjunctive 3VLPSN. Likewise, absence of *inferentially* consistent (viz., refuting the rule $p \vdash q$) non-subclassical extensions equally implies maximal paraconsistency.

As a matter of fact, the lattices of [axiomatic] extensions of LP, HZ, both $\mathbb{S}_3$ and its three-valued expansions [as well as P^1] have been found *ad hoc* in [15], [4], [19] [resp., [12]]. What has been especially remarkable in this connection is the similarity of the lattice of extensions of these logics (aside from P^1), according to which they [being subclassical] have a unique [non-classical consistent] proper extension, this being relatively axiomatized by the *Ex Contradictione Quodlibet* rule and defined by the direct products of their defining three-valued matrix and its two-valued submatrices, in which case this is [not] inconsistent [while the classical extension is relatively axiomatized by the *Modus Ponens* rule for the *material* implication]. This point has inevitably raised the question what does unify these miscellaneous instances. And, although the universal study [19] has unified practically all instances, the very first one — LP — has proved beyond its scopes because of the negative result given by Proposition 5.11 therein. Here, we obtain an effective algebraic criterion of the fact that any 3VLPSN with lattice conjunction and disjunction has the very such lattice of extensions. This positively subsumes all the instances mentioned above, and so has definitely unified them. On the other hand, such is not, generally speaking, the case for arbitrary (even both conjunctive and disjunctive) 3VLPSN. More precisely, we prove that any both conjunctive and disjunctive *classically-valued* (viz., with connectives having solely classical values, and so subclassical) 3VLPSN (including P^1) has infinitely many extensions, the classical one not being relatively axiomatized by the *Modus Ponens* rule for the *material* implication, that relatively axiomatized by the *Ex Contradictione Quodlibet* rule not being defined by the direct product of its defining matrix and the two-valued submatrix of this. In this way, the present general study has definitely justified the principal paradigm of universal logical investigations consisting in discovering uniform points behind particular results obtained originally *ad hoc*.

The rest of the paper is as follows. The exposition of the material is perfectly self-contained (of course, modulo very basic issues of Set and Lattice Theories, Universal Algebra and Mathematical Logic — including Model Theory — to be consulted in standard mathematical handbooks like [1, 7, 8] or fundamental papers like [5]). We entirely follow the standard conventions (most of which have become a part of logical and algebraic folklore constituting foundations of General Logic) adopted in [20], to Sections 2 and 3 of which the reader is referred just in case it is necessary. Section 2 is then to provide most general issues proving beyond the scopes of the mentioned study, those appearing therein being still briefly recalled for the sake of self-containity. Likewise, Section 3 is a brief summary of certain more specific advanced issues used in the paper. Further, Section 4 is comprehensive definitive semantic marking the framework of the present study. Finally, the rest of sections is devoted to the issue of extensions of 3VLPSN within its miscellaneous aspects and contexts.

2. General Mathematical background

2.1. **Set-theoretical background.** As usual (cf., e.g., [8]), natural numbers (including 0) are treated as ordinals (viz., sets of lesser natural numbers), the set of all them being denoted by ω, in which case, given any $N \subseteq \omega \ni n \neq 0$, we set $(N \div n) \triangleq \{\frac{m}{n} \mid m \in N\}$, while functions are viewed as binary relations with the left/right components of their elements as their arguments/values, respectively, but with standard (viz., left-|right-hand) writing functions|arguments, respectively, in which case though $(f \circ g)(a) = g(f(a))$, where f and g are functions with $(\mathrm{img}\, f) \subseteq (\mathrm{dom}\, g)$ and $a \in (\mathrm{dom}\, f) = \mathrm{dom}(f \circ g)$, whereas singletons are identified with their unique elements, unless any confusion is possible.

Likewise, given any set S (and any equivalence relation θ on it) {as well as any $T \subseteq S$}, let $\wp_{[(\setminus)\alpha]}(S)$ [where $\alpha \subseteq \omega$] be the set of all subsets of S [of cardinality ⟨not⟩ in α] (and $\nu_\theta \triangleq \{\langle s, \theta[\{s\}]\rangle \mid s \in S\}$ — the *natural function of* θ *on* S) {as well as (both $(T/\theta) \triangleq \nu_\theta[T]$ — the *quotient of* T *by* θ — and) $\chi^T_S \triangleq ((T \times \{1\}) \cup ((S \setminus T) \times \{0\}))$ — the *characteristic function of* T *in* S}. Next, any S-tuple (viz., a function with domain S) is normally written in the sequence form $\bar{t}$, its s-th component (viz., the value on argument $s \in S$) being written as t_s. Further, set $\Delta_S \triangleq \{\langle s, s\rangle \mid s \in S\}$, binary relations of such a kind being referred to as *diagonal*, and $S^+ \triangleq \bigcup_{i \in (\omega \setminus 1)} S^i$, elements of $S^* \triangleq (S^0 \cup S^+)$ being identified with ordinary finite tuples/"[comma separated] sequences". Then, any binary operation $\diamond$ on S determines the equally-denoted mapping $\diamond : S^+ \to S$ as follows: by induction on the length (viz., domain) l of any $\bar{a} \in S^{+/*}$, put:

$$((\diamond\bar{a})/\overleftarrow{\bar{a}}) \triangleq \begin{cases} a_0/\varnothing & \text{if } l = (1/0), \\ ((\diamond(\bar{a}{\upharpoonright}(l-1))) \diamond a_{l-1})/ & \\ (\{\langle 0, a_{l-1}\rangle\} \cup (\overleftarrow{(\bar{a}{\upharpoonright}(l-1))} \circ ((-1){\upharpoonright}(l \setminus 1)))) & \text{otherwise}/, \end{cases}$$

$\overleftarrow{\bar{a}}$ being the finite sequence *inverse to* $\bar{a}$. In particular, given any $f : S \to S$ and any $n \in \omega$, set $f^n \triangleq (\circ\langle n \times \{f\}, \Delta_S\rangle)$, in which case $f^1 = f$ and $f^0 = \Delta_S$. Finally, an *enumeration of* S is any bijection from its cardinality $|S|$ onto S.

2.2. **Algebraic background.** In general, to unify algebraic notations, unless otherwise specified, algebra[ic system]s [cf. [7]; (including logical matrices; cf. [5])] are denoted by capital Fraktur [resp. Calligraphic] letters, their underlying sets (viz., carriers) [resp., underlying algebras (viz., algebra reducts)] being denoted by corresponding capital Italic [resp., Fraktur] letters.

Let Σ be a *(propositional/sentential) language|signature* constituted by *(propositional/sentential)* {*primary*} *connectives* of finite arity to be viewed as function symbols. In that case, [given any (non-empty, unless Σ contains a nullary connective) $\alpha \subseteq \omega$] the absolutely-free Σ-algebra, freely generated by the set $V_{\omega[\cap\alpha]} \triangleq \{x_i \mid i \in (\omega[\cap\alpha])\}$ of *(propositional/sentential) variables [of rank* α*]*, is denoted by $\mathfrak{Fm}^{[\alpha]}_\Sigma$, the standard algebra superscript being normally omitted in writing its operations, "elements of its carrier $\mathrm{Fm}^{[\alpha]}_\Sigma$"|"its endomorphisms" being called *(propositional/sentential)* Σ*-formulas|-substitutions [of rank* α*]* "to be viewed as Σ-terms [of rank α]"|. Then, an *inverse* Σ*-substitution* is any function of the form $\{\langle X, \sigma^{-1}[X]\rangle \mid X \subseteq \mathrm{Fm}_\Sigma\}$, where σ is a Σ-substitution. Likewise, given any $n \in \omega$, a *secondary* n*-ary connective of* Σ is any Σ-formula of rank $\max(1, n)$, any primary one $\varsigma \in \Sigma$ being identified with the secondary one $\varsigma(\bar{x}_n)$, where $\bar{x}_n \triangleq \langle x_i\rangle_{i \in n}$.

As usual, elements of $\mathrm{Eq}_\Sigma^{[\alpha]} \triangleq (\mathrm{Fm}_\Sigma^{[\alpha]})^2$ are called Σ-*equations/-identities [of rank* α*]*, any $\langle\phi,\psi\rangle \in \mathrm{Eq}_\Sigma$ being normally written in the conventional equational form $\phi \approx \psi$. In this way, given any Σ-algebra $\mathfrak{A}$ and any $h \in \hom(\mathfrak{Fm}_\Sigma^{[\alpha]},\mathfrak{A})$, $(\ker h) \in \mathrm{Con}(\mathfrak{Fm}_\Sigma^{[\alpha]})$ is nothing but the set of all Σ-equations/-identities [of rank α] *true|satisfied in* $\mathfrak{A}$ *under* h. Likewise, given any class of Σ-algebras K, $\theta_\mathsf{K}^{[\alpha]} \triangleq (\mathrm{Eq}_\Sigma^{[\alpha]} \cap \bigcap\{\ker h \mid h \in \hom(\mathfrak{Fm}_\Sigma^{[\alpha]},\mathfrak{A}), \mathfrak{A} \in \mathsf{K}\}) \in \mathrm{Con}(\mathfrak{Fm}_\Sigma^{[\alpha]})$, for $\mathrm{Con}(\mathfrak{Fm}_\Sigma^{[\alpha]})$ is a closure system over $\mathrm{Eq}_\Sigma^{[\alpha]}$, is nothing but the set of all Σ-equations/-identities [of rank α] *true|satisfied in* K, in which case we set $\mathfrak{F}_\mathsf{K}^{[\alpha]} \triangleq (\mathfrak{Fm}_\Sigma^{[\alpha]}/\theta_\mathsf{K}^{[\alpha]})$. In case α as well as both K and all members of it are finite, the set $I \triangleq \{\langle\mathfrak{A},h\rangle \mid \mathfrak{A} \in \mathsf{K}, h \in \hom(\mathfrak{Fm}_\Sigma^{[\alpha]},\mathfrak{A})\}$ is finite, and so is $\mathfrak{F}_\mathsf{K}^{[\alpha]}$, for $e : F_\mathsf{K}^{[\alpha]} \to (\prod_{i\in I} A_i), \nu_\theta(a) \mapsto \langle h_i(a)\rangle_{i\in I}$, where, for all $i \in I$, $\mathfrak{A}_i \triangleq \pi_0(i)$ and $h_i \triangleq \pi_1(i)$, while $\theta \triangleq \theta_\mathsf{K}^{[\alpha]} \subseteq (\ker h_i)$, is an embedding of $\mathfrak{F}_\mathsf{K}^{[\alpha]}$ into the finite Σ-algebra $\prod_{i\in I}\mathfrak{A}_i$.

A *"congruence-permutation term"/discriminator for* K is any $\tau \in \mathrm{Fm}_\Sigma^3$ such that, for each $\mathfrak{A} \in \mathsf{K}$ and all $\bar{a} \in A^{2/3}$, it holds that $[\tau^\mathfrak{A}(a_0,a_1,a_{1/2}) =]a_0 = \tau^\mathfrak{A}(a_1,a_1,a_0)$ [unless $a_0 = a_1$] /"in which case it is a congruence-permutation term for $\mathfrak{A}$ (when taking $a_2 = a_1$)".

Given a Σ-algebra $\mathfrak{A}$, a subset $B \subseteq A$ is said to *"form a subalgebra of* $\mathfrak{A}$*"/"be* $\mathfrak{A}$*-closed"*, whenever $\varsigma^\mathfrak{A}[B^n] \subseteq B$, for all $\varsigma \in \Sigma$ of arity $n \in \omega$, in which case we have $(\varsigma^\mathfrak{A} \upharpoonright B) \triangleq (\varsigma^\mathfrak{A}\restriction B^n) : B^n \to B$, and so get the subalgebra $(\mathfrak{A}\restriction B) \triangleq \langle B, \varsigma^\mathfrak{A} \upharpoonright B\rangle_{\varsigma\in\Sigma}$ of $\mathfrak{A}$, called the *restriction of* $\mathfrak{A}$ *onto* B. Then, for any algebraic system $\mathcal{A}$ of a first-order signature $\Sigma\cup\Pi$ with underlying algebra $\mathfrak{A}$, we have the algebraic system $(\mathcal{A}\restriction B) \triangleq \langle\mathfrak{A}\restriction B, \rho^\mathcal{A} \upharpoonright B\rangle_{\rho\in\Pi}$, where $(\rho^\mathcal{A} \upharpoonright B) \triangleq (\rho^\mathcal{A} \cap B^n)$, for all $\rho \in \Pi$ of arity $n \in \omega$, called a/the *subsystem/restriction of* $\mathcal{A}$ *"/onto* B*"*.

Unless otherwise specified, throughout the paper, $\wr/(\diamond|\barwedge|\veebar|\sqsupset)$ is/are supposed to be a/ (possibly, secondary) unary/binary connective/s of Σ.

2.2.1. *Implicative systems.* According to [19], an *implicative system for* a class of Σ-algebras K is any $\mho \in \wp_\omega(\mathrm{Eq}_\Sigma^4)$ such that, for each $\mathfrak{A} \in \mathsf{K}$ and all $\bar{a} \in A^4$, it holds that $(a_0 = a_1) \Rightarrow (a_2 = a_3)$ iff $\mathfrak{A} \models (\bigwedge\mho)[x_i/a_i]_{i\in 4}$, in which case $\mho$ is an implicative system for every subalgebra of any member of K. Then, a quasivariety of Σ-algebras is said to be *implicative*, whenever it is generated by a subclass with an implicative system.

2.3. **Lattice-theoretical background.** A Σ-algebra $\mathfrak{A}$ is called a $\diamond$*-semi-lattice*, provided it satisfies semilattice (viz., idempotencity, commutativity and associativity) identities for $\diamond$, in which case we have the partial ordering $\leq_\diamond^\mathfrak{A}$ on A, given by $(a \leq_\diamond^\mathfrak{A} b) \overset{\text{def}}{\Longleftrightarrow} (a = (a \diamond^\mathfrak{A} b))$, for all $a, b \in A$. Then, in case the poset $\langle A, \leq_\diamond^\mathfrak{A}\rangle$ has the least element {viz., *zero*} [in particular, when A is finite], this is denoted by $\flat_\diamond^\mathfrak{A}$, while $\mathfrak{A}$ is referred to as a $\diamond$*-semi-lattice with zero (a)* (whenever $a = \flat_\diamond^\mathfrak{A}$).

A Σ-algebra $\mathfrak{A}$ is called a *[distributive]* $(\barwedge,\veebar)$*-lattice*, provided it satisfies [distributive] lattice identities for $\barwedge$ and $\veebar$ (viz., semilattice identities for both $\barwedge$ and $\veebar$ as well as mutual [both] absorption [and distributivity] identities for them), in which case $\leq_\barwedge^\mathfrak{A}$ and $\leq_\veebar^\mathfrak{A}$ are inverse to one another, and so, in case $\mathfrak{A}$ is a $\veebar$-semi-lattice with zero (in particular, when A is finite), $\flat_\veebar^\mathfrak{A}$ is the greatest element (viz., *unit*) of the poset $\langle A, \leq_\barwedge^\mathfrak{A}\rangle$. Then, in case $\mathfrak{A}$ is a {distributive} $(\barwedge,\veebar)$-lattice, it is said to be that *with zero/unit (a)*, whenever it is a $(\barwedge/\veebar)$-semilattice with zero (a).

As usual, [bounded] lattices are supposed to be of the signature $\Sigma_{+[,01]} \triangleq (\{\wedge,\vee\}[\cup\{\bot,\top\}])$ with binary $\wedge$ and $\vee$ [as well as nullary $\bot$ and $\top$]. Then, a *[bounded] (distributive) lattice* is any $\Sigma_{+[,01]}$-algebra $\mathfrak{A}$, being a (distributive) $(\wedge,\vee)$-lattice [with zero $\bot^\mathfrak{A}$ and unit $\top^\mathfrak{A}$] {cf., e.g., [1]}. Given any $n \in (\omega\setminus 2)$, by $\mathfrak{D}_{n[,01]}$

we denote the chain [bounded] distributive lattice with $D_{n[,01]} \triangleq (n \div (n-1))$ and $\leq^{\mathfrak{D}_{n[,01]}} = \leqslant$.

2.4. **Logical background.**

2.4.1. *Sentential calculi and logics.* Elements/subsets of $\wp_{[(\{\setminus\}1)\|\omega\|2]}(\mathrm{Fm}_\Sigma) \times \mathrm{Fm}_\Sigma$ are called *(propositional|sentential) [{non-}axiomatic||finitary||unary] Σ-rules/-calculi*, respectively, any [axiomatic] Σ-rule $\langle \Gamma, \varphi \rangle$ being normally written in the standard sequent form $\Gamma \vdash \varphi$ and semantically viewed as the infinitary basic Horn formula $(\bigwedge \Gamma) \to \varphi$ of the first-order signature $\Sigma \cup \{D\}$ with single unary *truth* predicate D — under the identification of any Σ-formula ψ with the atomic first-order formula $D(\psi)$ — [as well as being referred to as a *(propositional|sentential) Σ-axiom* and identified with φ].

Now, recall that a *(propositional/sentential) Σ-logic* (cf., e.g., [5]) is any closure operator C over Fm_Σ that is *structural* in the sense that $\sigma[C(X)] \subseteq C(\sigma[X])$, for all $X \subseteq \mathrm{Fm}_\Sigma$ and all $\sigma \in \hom(\mathfrak{Fm}_\Sigma, \mathfrak{Fm}_\Sigma)$, that is, the closure system $\operatorname{img} C$ over Fm_Σ is closed under inverse Σ-substitutions. Then, a Σ-rule $\Gamma \to \varphi$ is said to be *satisfied/derivable in* C, provided $\varphi \in C(\Gamma)$, Σ-axioms satisfied in C being called its *theorems*. Next, a Σ-logic C' is said to be a *[proper] extension of* C ($C \subseteq [\subsetneq] C'$, in symbols), provided $[C' \neq C$ and$]$ $C(X) \subseteq C'(X)$, for all $X \subseteq \mathrm{Fm}_\Sigma$, in which case C is referred to as a *[proper] sublogic of* C', while C' is the point-wise union $C \cup C'$. Then, C' is said to be *axiomatized by* a[n axiomatic] Σ-calculus $\mathcal{C}$ *(relatively to C)*, provided C' is the least {under the extension partial ordering ⊆} Σ-logic (being an extension of C and) satisfying every Σ-rule in $\mathcal{C}$ [(in which case C' is called an *axiomatic extension of* C, while

$$C'(X) = C(X \cup \bigcup\{\sigma[\mathcal{C}] \mid \sigma \in \hom(\mathfrak{Fm}_\Sigma, \mathfrak{Fm}_\Sigma)\}), \tag{2.1}$$

for all $X \subseteq \mathrm{Fm}_\Sigma$)]. Likewise, C' and C are said to be *([\]α)-equivalent* ($C' \equiv_{[\setminus]\alpha} C$, in symbols), where $\alpha \subseteq \omega$, provided $C(X) = C'(X)$, for all $X \in \wp_{[\setminus]\alpha}(\mathrm{Fm}_\Sigma)$. (In this connection, "[non-]axiomatically/finitely" stands for $([\setminus]1)/\omega$, respectively.) Then, a Σ-rule $\mathcal{R}$ is said to be *admissible in* C, provided the extension of C relatively axiomatized by $\mathcal{R}$ is axiomatically-equivalent to C. Clearly, $\mathcal{R}$ is admissible in C, whenever it is derivable in it. If the converse holds in general (i.e., any Σ-rule is derivable in C, whenever it is admissible in it), then C is said to be *structurally/deductively complete|maximal*. Clearly, C is structurally complete iff it has no proper axiomatically-equivalent extension. Then, the set S of all Σ-logics axiomatically-equivalent to C contains C itself, in which case the closure system $(\bigcap\{\operatorname{img} C' \mid C' \in S\}) \ni C(\varnothing)$ over Fm_Σ is closed under inverse Σ-substitutions, and so the corresponding closure operator over Fm_Σ is the greatest Σ-logic axiomatically-equivalent to C, being then a structurally complete extension of C, called the *structural completion of* C. Likewise, we have the greatest finitary sublogic $C_\dashv$ of C, defined by $C_\dashv(X) \triangleq (\bigcup C[\wp_\omega(X)])$, for all $X \subseteq \mathrm{Fm}_\Sigma$, being then finitely-equivalent to C and called the *finitarization of* C. Then, the extension of any finitary (in particular, diagonal) Σ-logic relatively axiomatized by a finitary Σ-calculus is a sublogic of its own finitarization, in which case it is equal to this, and so is finitary (in particular, the Σ-logic axiomatized by a finitary Σ-calculus is finitary; conversely, any [finitary] Σ-logic is axiomatized by the [finitary] Σ-calculus consisting of all those [finitary] Σ-rules, which are satisfied in C). Further, C is said to be *[strongly]/weakly ⋄-conjunctive|-disjunctive*, provided $C(X \cup \{\phi \diamond \psi\}) = /(\supseteq \mid \subseteq) C(C(X \cup \{\phi\})(\cup|\cap) C(X \cup \{\psi\}))$, for all $X \subseteq \mathrm{Fm}_\Sigma$ and all $\phi, \psi \in \mathrm{Fm}_\Sigma$ |, "in which case"/"that is, the first two (viz., (2.2) with $i \in 2$) of" the following rules:

$$x_i \quad \vdash \quad (x_0 \diamond x_1), \tag{2.2}$$

$$(x_0 \diamond x_1) \quad \vdash \quad (x_1 \diamond x_0), \tag{2.3}$$

$$(x_0 \diamond x_0) \quad \vdash \quad x_0, \tag{2.4}$$

where $i \in 2$, are satisfied in C, and so in its extensions. Likewise, C is said to be *weakly $\diamond$-implicative*, provided it satisfies the *Modus Ponens* rule:

$$\{x_0, x_0 \diamond x_1\} \to x_1 \tag{2.5}$$

and *has Deduction* (viz,. *Herbrand*; cf. [8]) *theorem (DT/HT) with respect to* $\diamond$ in the sense that, for all $\phi \in X \subseteq \mathrm{Fm}_\Sigma$ and all $\psi \in C(X)$, it holds that $(\phi \diamond \psi) \in C(X \setminus \{\phi\})$, in which case the following axioms:

$$(x_0 \diamond x_0), \tag{2.6}$$

$$(x_0 \diamond (x_1 \diamond x_0) \tag{2.7}$$

are satisfied in C. Then, C is said to be *(strongly) $\diamond$-implicative*, whenever it is weakly so and satisfies the *Peirce Law* axiom (cf. [9]):

$$(((x_0 \diamond x_1) \diamond x_0) \diamond x_0). \tag{2.8}$$

Furthermore, C is said to be *($\{$axiomatically$\}$ $\langle$pre$\rangle$maximally) [inferentially-|$\wr$-para]consistent*, provided $x_1 \notin C(\varnothing[\cup\{\wr^k x_0 \mid k \in (1|2)\}])$ (and C has no $\langle$more than one$\rangle$ proper [inferentially-|$\wr$-para]consistent $\{$axiomatic$\}$ extension). Then, by C^{IC} [resp., C^{NP}] we denote the least non-[$\wr$-para]consistent extension of C, that is, the one relatively axiomatized by x_0 [resp., the *Ex Contradictione Quodlibet* rule:

$$\{x_0, \wr x_0\} \vdash x_1] \tag{2.9}$$

Likewise, C is said to be *$\diamond$-implicatively $\wr$-paraconsistent*, provided it does not satisfy the *Ex Contradictione Quodlibet* axiom:

$$\wr x_0 \diamond (x_0 \diamond x_1). \tag{2.10}$$

(Clearly, C is non-$\wr$-paraconsistent if[f] it is $\diamond$-implicatively so, whenever it satisfies (2.5) [and has HT with respect to $\diamond$].) In general, by C^{INP} we denote the axiomatic extension of C relatively axiomatized by (2.10). Furthermore, $(\mathrm{img}\, C) \cup \{\varnothing\}$ is a closure system over Fm_Σ closed under inverse Σ-substitutions, the corresponding closure operator C_{+0} over Fm_Σ being the greatest sublogic of C without theorems as well as non-axiomatically-equivalent to C. Finally, given any $\Sigma' \subseteq \Sigma$, we have the Σ'-logic C', given by $C'(X) \triangleq (C(X) \cap \mathrm{Fm}_{\Sigma'})$, for all $X \subseteq \mathrm{Fm}_{\Sigma'}$, called the *$\Sigma'$-fragment of C*, in which case C is referred to as a *(Σ-)expansion of C'*.

Remark 2.1. Given any Σ-logic C without theorems, $(C^{\mathrm{IC}})_{+0}$ is the structural completion of C and is consistent (as it has no theorem) but inferentially-inconsistent, for it is non-axiomatically-equivalent to the inconsistent C^{IC}. In particular, any Σ-logic without theorems is not structurally complete, unless it is inferentially-inconsistent. □

2.4.2. *Logical matrices.* As usual, any *(logical) Σ-matrix* $\mathcal{A} = \langle \mathfrak{A}, D^{\mathcal{A}} \rangle$ with its *underlying* Σ-algebra $\mathfrak{A}$, elements of A being viewed as *values of* $\mathcal{A}$, and its *truth predicate* (viz., the set of its *distinguished* values) $D^{\mathcal{A}} \subseteq A$ (cf., e.g., [5], to which the reader is referred for the conception of the *logic* $\mathrm{Cn}_{\mathcal{A}}$ *of/"defined by"* $\mathcal{A}$, the *logic* Cn_{M} *of/"defined by"* a class of Σ-matrices M being then the "point-wise" intersection of the logics of all members of M) is treated as a first-order model structure (viz, an algebraic system; cf. [7], to which the reader is referred for notions of [sub]direct product|power|square [as a subsystem of the direct one with surjective projections], "isomorphism as that between underlying algebras preserving relations"/"embedding as an isomorphism onto a subsystem" between systems, "isomorphic as having an isomorphism"/"embeddable as isomorphic to a subsystem" systems, etc.) of the first-order signature $\Sigma \cup \{D\}$, in which case the "underlying algebra"/"truth predicate" of the direct product|power| square of a tuple of Σ-matrices is just the direct product|power|square of the "underlying algebras"/"truth predicates" of the tuple's components, while any Σ-rule is *true/satisfied in* $\mathcal{A}$ iff it is satisfied in $\mathrm{Cn}_{\mathcal{A}}$. Then, $\mathcal{A}$ is said to be *finite[ly-generated]/"generated by* $B \subseteq A$*"|n-valued*, where $n > 0$, whenever $\mathfrak{A}$ is so|n-element, respectively, the logics of n-valued Σ-matrices being well-known to be finitary (cf., e.g., [5]) and referred to as *n-valued*. Next, $\mathcal{A}$ is said to be *truth-|false-singular/truth-[non-]empty*, provided $(|D^{\mathcal{A}}|(A \setminus D^{\mathcal{A}})| \in 2)/(D^{\mathcal{A}} \in \wp_{[\setminus]1}(A))$, respectively. Furthermore, $\mathcal{A}$ is said to be *[strongly]/weakly* $\diamond$*-conjunctive|-disjunctive*, provided, for all $a, b \in A$, $(a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}}$ iff/""only if"|if" both|either $a \in D^{\mathcal{A}}$ and|or $b \in D^{\mathcal{A}}$, "that is"|""in which case"/"that is"" its logic is so, respectively. Likewise, $\mathcal{A}$ is said to be $\diamond$*-implicative*, provided, for all $a, b \in A$, $(a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}}$ iff either $a \notin D^{\mathcal{A}}$ or $b \in D^{\mathcal{A}}$, in which case it is $\vee_\diamond$-disjunctive, where $(x_0 \vee_\diamond x_1) \triangleq ((x_0 \diamond x_1) \diamond x_1)$, and so its logic is strongly $\diamond$-implicative, for $(2.8) = ((x_0 \diamond x_1) \vee_\diamond x_0)$. Next, $\mathcal{A}$ is said to be *[inferentially-|$\wr$-para]consistent*, provided $A \neq D^{\mathcal{A}}$ [and $(\wr^{\mathfrak{A}})^k a \in D^{\mathcal{A}}$, for some $a \in A$ and all $k \in (1|2)$], that is, the logic of it is so. Likewise, $\mathcal{A}$ is said to be $\wr$*-negative*, provided $(a \in D^{\mathcal{A}}) \Leftrightarrow (\wr^{\mathfrak{A}} a \notin D^{\mathcal{A}})$, for all $a \in A$, in which case it is inferentially-consistent but is not $\wr$-paraconsistent, while it is [weakly] $\diamond$-conjunctive/-disjunctive iff it is [weakly] $\diamond^{\wr}$-disjunctive/-conjunctive, where $(x_0 \diamond^{\wr} x_1) \triangleq \wr(\wr x_0 \diamond \wr x_1)$, whereas it is $\sqsupset^{\wr}_{\diamond}$-implicative, whenever it is $\diamond$-disjunctive, where $(x_0 \sqsupset^{\wr}_{\diamond} x_1) \triangleq (\wr x_0 \diamond x_1)$ is the *material* implication. Furthermore, according to [15], a set ∇ of Σ-equations of rank 1 is said to *define (equationally) truth [predicate] of/in* $\mathcal{A}$, provided, for all $a \in A$, $a \in D^{\mathcal{A}}$ iff $\mathfrak{A} \models (\bigwedge \nabla)[x_0/a]$. Further, a *congruence of* $\mathcal{A}$ is any $\theta \in \mathrm{Con}(\mathfrak{A})$ such that $\theta \subseteq (\ker \chi^{\mathcal{A}})$ — the *characteristic relation of* $\mathcal{A}$, where $\chi^{\mathcal{A}} \triangleq \chi_A^{D^{\mathcal{A}}}$ is the *characteristic function of* $\mathcal{A}$ (in which case we have the *quotient* Σ-matrix $(\mathcal{A}/\theta) \triangleq \langle \mathfrak{A}/\theta, D^{\mathcal{A}}/\theta \rangle$), the set of all them being denoted by $\mathrm{Con}(\mathcal{A}) \ni \Delta_A$, $\mathcal{A}$ being said to be *[(finitely) hereditarily] simple*, whenever it has no non-diagonal congruence [as well as no non-simple (finitely-generated) submatrix]. Next, $\mathcal{A}$ is said to be a *model of* a Σ-logic C [*over* $\mathfrak{A}$], provided its logic is an extension of C, the class [viz., set] of all them being denoted by $\mathrm{Mod}^{[\mathfrak{A}]}(C)$. Then, $\pi_1[\mathrm{Mod}^{\mathfrak{Fm}^{\omega}_{\Sigma}}(C) = (\mathrm{img}\, C)$, in view of the structurality of C. Further, both two-valued and $\wr$-negative Σ-matrices (with diagonal characteristic function) are said to be *(canonically)* $\wr$*-classical*, [{proper} sublogics of] their logics being referred to as *[{properly}]* $\wr$*-[sub]classical*. ⟨Clearly, any $\wr$-classical Σ-matrix is isomorphic to a canonically $\wr$-classical one, while any isomorphic canonically $\wr$-classical Σ-matrices are equal, for isomorphisms between them are diagonal.⟩ Likewise, a unary $\sim \in \Sigma$ (fixed throughout the paper by default as negation) is referred to as a *subclassical negation for* a Σ-logic C, whenever the $\sim$-fragment of C is $\sim$-subclassical, in which case:

$$\sim^m x_0 \notin C(\sim^n x_0), \tag{2.11}$$

for all $m, n \in \omega$ such that the integer $m - n$ is odd. Finally, given any $\Sigma' \subseteq \Sigma$, $\mathcal{A}$ is said to be a *(Σ-)expansion of* $(\mathcal{A}{\restriction}\Sigma') \triangleq \langle \mathfrak{A}{\restriction}\Sigma', D^{\mathcal{A}} \rangle$, then defining the Σ'-fragment of the logic of $\mathcal{A}$.

Given Σ-matrices $\mathcal{A}$ and $\mathcal{B}$ such that the set $\hom^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A}, \mathcal{B}) \triangleq \{h \in \hom(\mathfrak{A}, \mathfrak{B}) \mid [h[A] = B,] D^{\mathcal{A}} \subseteq h^{-1}[D^{\mathcal{B}}] (\subseteq D^{\mathcal{A}})\}$ of all *(strict) [surjective] homomorphisms from* $\mathcal{A}$ *[on]to* $\mathcal{B}$, injective/bijective strict homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ being exactly embeddings/isomorphisms of/from $\mathcal{A}$ into/onto $\mathcal{B}$, $\mathcal{A}$ being a submatrix of $\mathcal{B}$ iff $\Delta_A \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, is not empty (in which case $\mathcal{A}$ is {weakly} $\wr$-negative/

$\diamond$-conjunctive|-disjunctive|-implicative if[f] $\mathcal{B}$ is so, while the logic of $\mathcal{A}$ is a [non-proper] extension of the one of $\mathcal{B}$; cf. (2.2) of [20], whereas $(\ker h) \in \mathrm{Con}(\mathcal{A})$, and so h is injective, whenever $\mathcal{A}$ is simple; cf. Remark 2.2 and Corollary 2.3 of [20]) [in which case theorems of the logic of $\mathcal{A}$ are those of $\mathcal{B}$; cf. (2.3) of [20]], $\mathcal{B}|\mathcal{A}$ is referred to as a *(strict) [surjective] homomorphic image|counter-image of* $\mathcal{A}|\mathcal{B}$, respectively. Then, the class of all "(consistent) submatrices of"/"strict surjective homomorphic [counter-]images of"/"Σ-matrices isomorphic to" members of any class M of Σ-matrices is denoted by $(\mathbf{S}_{(*)}/\mathbf{H}^{[-1]}/\mathbf{I})(\mathsf{M})$, respectively. Likewise, the class of all [sub]direct products of {finite} tuples constituted by members of M is denoted by $\mathbf{P}^{[\mathrm{SD}]}_{\{\omega\}}(\mathsf{M})$, respectively.

Lemma 2.2 (Finite Subdirect Product Lemma; cf. Lemma 2.7 of [20])**.** *Let* M *be a finite class of finite* Σ*-matrices and* $\mathcal{A}$ *a finitely-generated (in particular, finite) model of the logic of* M*. Then,* $\mathcal{A} \in \mathbf{H}^{-1}(\mathbf{H}(\mathbf{P}^{\mathrm{SD}}_{\omega}(\mathbf{S}_*(\mathsf{M}))))$*.*

Theorem 2.3 (cf. Theorem 2.8 of [20])**.** *Let* K *and* M *be classes of* Σ*-matrices,* C *the logic of* M *and* C' *an extension of* C*. Suppose [both* M *and all members of it are finite and]* $\mathbf{P}^{\mathrm{SD}}_{[\omega]}(\mathbf{S}_*(\mathsf{M})) \subseteq \mathsf{K}$ *(in particular,* $\mathbf{S}(\mathbf{P}_{[\omega]}(\mathsf{M})) \subseteq \mathsf{K}$ *{in particular,* $\mathsf{K} \supseteq \mathsf{M}$ *is closed under both* $\mathbf{S}$ *and* $\mathbf{P}_{[\omega]}\langle$ *in particular,* $\mathsf{K} = \mathrm{Mod}(C)\rangle\}$*). Then,* C' *is [finitely-equivalent to the logic] defined by* $\mathrm{Mod}(C') \cap \mathsf{K}$*.*

Corollary 2.4 (cf. Corollary 2.9 of [20])**.** *Let* M *be a class of* Σ*-matrices and* $\mathcal{A}$ *an axiomatic* Σ*-calculus. Then, the axiomatic extension of the logic of* M *relatively axiomatized by* $\mathcal{A}$ *is defined by* $\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{A})$*.*

3. Preliminary key generic issues

3.1. Peculiarities of false-singular matrices.

3.1.1. *Conjunctive matrices.*

Lemma 3.1. *Let* $\mathcal{A}$ *be a false-singular weakly* $\diamond$*-conjunctive* Σ*-matrix,* $f \in (A \setminus D^{\mathcal{A}})$*,* I *a finite set,* $\overline{\mathcal{B}}$ *an* I*-tuple constituted by consistent submatrices of* $\mathcal{A}$ *and* $\mathcal{D}$ *a subdirect product of it. Then,* $(I \times \{f\}) \in D$*.*

Proof. By induction on the cardinality of any $J \subseteq I$, let us prove that there is some $a \in D$ including $(J \times \{f\})$. First, when $J = \varnothing$, take any $a \in D \neq \varnothing$, in which case $(J \times \{f\}) = \varnothing \subseteq a$. Now, assume $J \neq \varnothing$. Take any $j \in J \subseteq I$, in which case $K \triangleq (J \setminus \{j\}) \subseteq I$, while $|K| < |J|$, and so, as $\mathcal{B}_j$ is a consistent submatrix of the false-singular Σ-matrix $\mathcal{A}$, we have $f \in B_j = \pi_j[D]$. Hence, there is some $b \in D$ such that $\pi_j(b) = f$, while, by induction hypothesis, there is some $a \in D$ including $(K \times \{f\})$. Therefore, since $J = (K \cup \{j\})$, while $\mathcal{A}$ is both weakly $\diamond$-conjunctive and false-singular, we have $D \ni c \triangleq (a \diamond^{\mathfrak{D}} b) \supseteq (J \times \{f\})$. Thus, when $J = I$, we eventually get $D \ni (I \times \{f\})$, as required. □

3.1.2. *Disjunctive matrices.*

Lemma 3.2. *Let* $\mathcal{A}$ *be a false-singular* Σ*-matrix and* C *the logic of it. Then, the following are equivalent:*

(i) C *is* $\diamond$*-disjunctive;*
(ii) C *(viz.,* $\mathcal{A}$*) satisfies* (2.2) *with* $i = 0$*,* (2.3) *and* (2.4)*;*
(iii) $\mathcal{A}$ *is* $\diamond$*-disjunctive.*

Proof. First, (iii/i)$\Rightarrow$(i/ii) are immediate. Finally, assume (ii) holds. Consider any $a, b \in A$. Then, in case either of $(a/b) \in D^{\mathcal{A}}$ holds, by (2.2) with $i = 0$ /"and (2.3)", we have $(a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}}$. Otherwise, $a = b$, and so, by (2.4), we get $D^{\mathcal{A}} \not\ni (a \diamond^{\mathfrak{A}} a) = (a \diamond^{\mathfrak{A}} b)$. Thus, (iii) holds, as required. □

3.1.3. *Implicative matrices.*

Lemma 3.3. *Let $\mathcal{A}$ be a false-singular Σ-matrix and C the logic of it. Then, the following are equivalent:*

(i) *C is stronly $\diamond$-implicative;*
(ii) *C is weakly $\diamond$-implicative;*
(iii) *C (viz., $\mathcal{A}$) satisfies* (2.6), (2.7) *and* (2.5)*;*
(iv) *$\mathcal{A}$ is $\diamond$-implicative.*

Proof. First, (iv/ii)⇒(i/iii) are immediate. Next, (ii) is a particular case of (i). Finally, assume (iii) holds. Consider any $a, b \in A$. Then, by (2.5) and (2.7), $(a \diamond^{\mathfrak{A}} b) \in / \notin D^{\mathcal{A}}$, whenever $b \in / \notin D^{\mathcal{A}} / \ni a$. Now, assume $a \notin D^{\mathcal{A}} \not\ni b$, in which case $a = b$, and so, by (2.6), $D^{\mathcal{A}} \ni (a \diamond^{\mathfrak{A}} a) = (a \diamond^{\mathfrak{A}} b)$. Thus, (iv) holds. □

3.2. **Disjunctivity and non-paraconsistency versus Resolution.** Given any Σ-logic C, by C^{R} we denote the extension of C relatively axiomatized by the *Resolution* rule (cf. [21] for roots of such terminology):

$$\{x_0 \veebar x_1, \wr x_0 \veebar x_1\} \vdash x_1. \tag{3.1}$$

Likewise, by C^{MP} we denote the extension of C relatively axiomatized by the *Modus Ponens* rule (2.5) for the material implication $\diamond = \sqsupset^{\sim}_{\veebar}$:

$$\{x_0, {\sim} x_0 \veebar x_1\} \vdash x_1, \tag{3.2}$$

being a sublogic/extension of $C^{\mathrm{R/NP}}$, whenever C is weakly $\veebar$-disjunctive.

Lemma 3.4. (3.1) *is satisfied in any $\veebar$-disjunctive non-$\wr$-paraconsistent Σ-logic C.*

Proof. In that case, we have $x_1 \in (C(x_1) \cap C(\{x_0, \wr x_0\})) = (C(x_1) \cap C(\{x_0 \veebar x_1, \wr x_0\})) = C(\{x_0 \veebar x_1, \wr x_0 \veebar x_1\})$, as required. □

Given a class M of Σ-matrices, by $\mathbf{S}^{\mathrm{NP}}_{*}(\mathsf{M})$ we denote the class of all non-$\wr$-paraconsistent members of $\mathbf{S}_{*}(\mathsf{M})$.

Theorem 3.5. *Let M be a finite class of finite $\veebar$-disjunctive Σ-matrices and C the logic of M. Then, C^{R} is defined by $\mathbf{S}^{\mathrm{NP}}_{*}(\mathsf{M})$, and so is $\veebar$-disjunctive.*

Proof. In that case, the logic of $\mathbf{S}^{\mathrm{NP}}_{*}(\mathsf{M})$ is a both $\veebar$-disjunctive and non-$\wr$-paraconsistent extension of C, and so an extension of C^{R}, in view of Lemma 3.4. Conversely, consider any $n \in (\omega \setminus 1)$ and any $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}^n_{\Sigma}$ such that $\varphi \notin C^{\mathrm{R}}(\Gamma)$. Then, as n as well as both $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and all members of it are finite, $\mathfrak{F}^n_{\Sigma}$, is finite. In particular, its subset P/θ, where $\theta \triangleq \theta^n_{\mathsf{K}}$ and $P \triangleq \{\phi \in \mathrm{Fm}^n_{\Sigma} \mid h[\{\phi, \wr\phi\}] \subseteq D^{\mathcal{A}}, \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^n_{\Sigma}, \mathfrak{A})\}$, is finite. Take any enumeration $\bar{e}$ of P/θ. For each $i \in m \triangleq |P/\theta| \in \omega$, choose any $\psi_i \in P$ such that $\nu_\theta(\psi_i) = e_i$. By induction on any $l \in (m+1)$, set

$$\Xi_l \triangleq \begin{cases} \{\varphi\} & \text{if } l = 0, \\ \{\wr^k \psi_{l-1} \veebar \phi \mid k \in 2, \phi \in \Xi_{l-1}\} & \text{otherwise,} \end{cases}$$

and prove that

$$\varphi \in C^{\mathrm{R}}(\Xi_l). \tag{3.3}$$

The case, when $l = 0$, is evident. Otherwise, by the induction hypothesis, (3.1) and the structurality of C^{R}, we have $\varphi \in C^{\mathrm{R}}(\Xi_{l-1}) \subseteq C^{\mathrm{R}}(\Xi_l)$. In particular, by (3.3) with $l = m$, we get $\mathrm{Fm}^n_{\Sigma} \supseteq \Xi_m \nsubseteq C^{\mathrm{R}}(\Gamma) \supseteq C(\Gamma)$. Hence, there are some $\mathcal{A} \in \mathsf{M}$ and some $h \in \hom(\mathfrak{Fm}^n_{\Sigma}, \mathfrak{A})$ such that $h[\Xi_m] \nsubseteq D^{\mathcal{A}} \supseteq h[\Gamma]$, in which case h is a homomorphism from $\mathfrak{Fm}^n_{\Sigma}$ onto the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $h[V_n]$. Let us prove, by contradiction, that $\mathcal{B} \triangleq \langle \mathfrak{B}, B \cap D^{\mathcal{A}} \rangle$ is not $\wr$-paraconsistent. For suppose $\mathcal{B}$ is $\wr$-paraconsistent. Then, there is some $\phi \in \mathrm{Fm}^n_{\Sigma}$ such that $h[\{\phi, \wr\phi\}] \subseteq D^{\mathcal{A}}$, in which case $\phi \in P$, and so there is some $i \in m$ such that $\phi\ \theta\ \psi_i$. In particular, as $\theta \subseteq (\ker h)$, $h([\wr]\psi_i) = h([\wr]\phi) \in D^{\mathcal{A}}$. Therefore, by the $\veebar$-disjunctivity of $\mathcal{A}$, we have $h[\Xi_{i+1}] \subseteq D^{\mathcal{A}}$, in which case, for all $j \in (m \setminus i)$, we get $h[\Xi_{j+1}] \subseteq D^{\mathcal{A}}$, and so, in particular (when $j = (m-1)$), we eventually get $h[\Xi_m] \subseteq D^{\mathcal{A}}$. This contradiction shows that $\mathcal{B}$ is not $\wr$-paraconsistent. Likewise, $h(\varphi) \notin D^{\mathcal{A}}$, because, otherwise, by the $\veebar$-disjunctivity of $\mathcal{A}$, we would have $h[\Xi_l] \subseteq D^{\mathcal{A}}$, for all $l \in (m+1)$, and so, in particular (when $l = m$), we would get $h[\Xi_m] \subseteq D^{\mathcal{A}}$. In this way, $\Gamma \vdash \varphi$ is not true in $\mathcal{B} \in \mathbf{S}_*^{\mathrm{NP}}(\mathsf{M})$ under $h \in \hom(\mathfrak{Fm}_\Sigma^n, \mathfrak{B})$, as required, for $\wp_\omega(\mathrm{Fm}_\Sigma^\omega) \subseteq \bigcup_{n\in\omega} \wp(\mathrm{Fm}_\Sigma^n)$, while the logic of $\mathbf{S}_*^{\mathrm{NP}}(\mathsf{M})$ is finitary. □

This has found applications, in particular, to four-valued logics studied in [20].

3.3. **Equality determinants.** Following the spirit of [17] and [18], an *equality determinant for* a class of Σ-matrices M is any infinitary quantifier-free equality-free formula Φ of the first-order signature $\Sigma \cup \{D\}$ with variables in V_2 such that the infinitary universal sentence $\forall x_0 \forall x_1 (\Phi \leftrightarrow (x_0 \approx x_1))$ with equality is true in M, in which case Φ is an equality determinant for $\mathbf{I}(\mathbf{S}(\mathsf{M}))$ (cf. Lemma 3.3 of [20] for the "unitary" case specified below). Then, a *canonical equality determinant for* M is any Σ-calculus $\varepsilon \subseteq (\wp(\mathrm{Fm}_\Sigma^2) \times \mathrm{Fm}_\Sigma^2)$ such that $\bigwedge \varepsilon$ is an equality determinant for M. Likewise, a *[canonical] unitary equality determinant for* K is any $\Upsilon \subseteq (\mathrm{Fm}_\Sigma^1[\setminus V_1])$ such that $\varepsilon^{\Upsilon[\dot{+}]} \triangleq \{(\upsilon[x_0/x_i]) \vdash (\upsilon[x_0/x_{1-i}]) \mid i \in 2, \upsilon \in (\Upsilon[\cup V_1])\}$ is a canonical equality determinant for M. It is unitary equality determinants that are equality determinants in the sense of [17] and [20]. Clearly, $\varnothing$ is a canonical unitary equality determinant for any inferentially-consistent two-valued (in particular, $\wr$-classical) Σ-matrix (cf. Example 1/3.1 of [17]/[20]), because its characteristic relation is diagonal, in which case it is hereditarily simple, for any one-valued Σ-matrix has a diagonal characteristic relation, and so is simple. And what is more, we have, in general:

Lemma 3.6 (cf. Lemma 3.2 of [20] for the "unitary" case)**.** *Any Σ-matrix $\mathcal{A}$ with equality determinant Φ is simple, and so hereditarily simple.*

Proof. Then, for any $\bar{a} \in \theta \in \mathrm{Con}(\mathcal{A})$, and all $\varphi \in \mathrm{Fm}_\Sigma^2$, we have $\varphi^{\mathfrak{A}}(a_0, a_0)\ \theta\ \varphi^{\mathfrak{A}}(a_0, a_1)$, in which case we get $(\varphi^{\mathfrak{A}}(a_0, a_0) \in D^{\mathcal{A}}) \Leftrightarrow (\varphi^{\mathfrak{A}}(a_0, a_1) \in D^{\mathcal{A}})$, and so $\mathcal{A} \models \Phi[x_i/a_i]_{i\in 2}$, for $\mathcal{A} \models \Phi[x_i/a_0]_{i\in 2}$, as $a_0 = a_0$ (in particular, $a_0 = a_1$, in which case $\theta = \Delta_A$, and so $\mathcal{A}$ is simple). □

Remark 3.7. Given an $\sqsupset$-implicative Σ-matrix $\mathcal{A}$ with a finitary canonical equality determinant ε, $\varepsilon_\sqsupset \triangleq \{\sqsupset \overleftarrow{\langle \bar{\phi}, \psi \rangle} \mid \bar{\phi} \in (\mathrm{Fm}_\Sigma^2)^*, \psi \in \mathrm{Fm}_\Sigma^2, \langle \mathrm{img}\, \bar{\phi}, \psi \rangle \in \varepsilon\}$ is an axiomatic canonical equality determinant for $\mathcal{A}$. □

Perhaps, a most distinctive feature of axiomatic canonical determinants is as follows:

Remark 3.8. Any $\varepsilon \subseteq \mathrm{Fm}_\Sigma^2$ is an axiomatic canonical equality determinant for a class M of Σ-matrices iff the following infinitary universal Horn sentences of the first-order signature $\Sigma \cup \{D\}$ with equality:

$$\forall x_0 \forall x_1 ((\bigwedge \varepsilon) \to (x_0 \approx x_1)), \tag{3.4}$$

$$\forall x_0 (\xi[x_1/x_0]), \tag{3.5}$$

where $\xi \in \varepsilon$, are true in M. On the other hand, model classes of infinitary universal Horn theories with equality are well-known to be closed under adjoining direct products and isomorphic copies as well as subsystems of their members. (cf., e.g., [7] for the "first-order" case immediately extended to the infinitary one). Thus, in particular, by Lemma 3.6, given a class M of Σ-matrices with a same axiomatic canonical equality determinant, every member of $\mathbf{I}(\mathbf{S}(\mathbf{P}(\mathsf{M})))$ is simple. □

On the other hand, due to Łoś-Mal'cev Compactness Theorem for classes of algebraic systems closed under ultra-products (cf., e.g., [7]) as well as the well-known "non-purely-agebraic" extension of Corollary 2.3 of [2] to algebraic systems, any finite class of finite Σ-matrices with a same canonical equality determinant has a finitary one. It appears such is the case for *arbitrary* classes of *arbitrary* Σ-matrices with *arbitrary* equality determinants. More precisely, as a consequence of Mal'cev's Principal Congruence Lemma [6], we have:

Theorem 3.9. *Let $\mathcal{A}$ be a Σ-matrix. Then, the following are equivalent:*

(i) *$\mathcal{A}$ is finitely hereditarily simple;*
(ii) *$\mathcal{A}$ is hereditarily simple;*
(iii) *$\mathcal{A}$ has an equality determinant;*
(iv) *$\mathcal{A}$ has a canonical equality determinant;*
(v) *$\mathcal{A}$ has a finitary canonical equality determinant;*
(vi) *$\mathcal{A}$ has a unary canonical equality determinant;*
(vii) *$\mathcal{A}$ has a unary non-axiomatic canonical equality determinant;*
(viii) $\varepsilon^2_\Sigma \triangleq \{\phi_i \vdash \phi_{1-i} \mid i \in 2, \bar{\phi} \in (\mathrm{Fm}^2_\Sigma)^2, (\phi_0[x_1/x_0]) = (\phi_1[x_1/x_0])\}$ *is a canonical equality determinant for $\mathcal{A}$.*

Proof. First, (iii)⇒(ii) is by Lemma 3.6. Next, (i/iii/iv/v/vi/vii) is a particular case of (ii/iv/v/vi/vii/viii), respectively.

Finally, assume (i) holds. Clearly, $\mathcal{A}(\bigwedge \varepsilon^2_\Sigma)[x_i/a]_{i\in 2}$, for all $a \in 2$. Conversely, consider any $\bar{a} \in (A^2 \setminus \Delta_A)$. Let $\mathcal{B}$ be the submatrix of $\mathcal{A}$ generated by the finite set $\operatorname{img}\bar{a}$. Then, it is simple, by (i), in which case the least congruence $\theta \ni \bar{a} \notin \Delta_B$ of $\mathfrak{B}$ is non-diagonal, and so $\theta \nsubseteq \theta^{\mathcal{B}}$. On the other hand, by Mal'cev's Principal Congruence Lemma [6], θ is the transitive closure of $\vartheta \triangleq \{\langle \varphi^{\mathfrak{B}}[x_0/a_j; x_{k+1}/c_k]_{k\in n}, \varphi^{\mathfrak{B}}[x_0/a_{1-j}; x_{k+1}/c_k]_{k\in n}\rangle \mid j \in 2, n \in \omega, \varphi \in \mathrm{Fm}^{n+1}_\Sigma, \bar{c} \in B^n\}$. Therefore, $\theta^{\mathcal{B}}$, being transitive, does not include ϑ, in which case there are some $j \in 2$, some $n \in \omega$, some $\varphi \in \mathrm{Fm}^{n+1}_\Sigma$ and some $\bar{c} \in B^n$ such that $\langle \varphi^{\mathfrak{B}}[x_0/a_j; x_{k+1}/c_k]_{k\in n}, \varphi^{\mathfrak{B}}[x_0/a_{1-j}; x_{k+1}/c_k]_{k\in n}\rangle \notin \theta^{\mathcal{B}}$, and so there is some $i \in 2$ such that $\varphi^{\mathfrak{B}}[x_0/a_i; x_{k+1}/c_k]_{k\in n} \in D^{\mathcal{B}} \not\ni \varphi^{\mathfrak{B}}[x_0/a_{1-i}; x_{k+1}/c_k]_{k\in n}$, while, as $\mathfrak{B}$ is generated by $\operatorname{img}\bar{a}$, there is some $\bar{\psi} \in (\mathrm{Fm}^2_\Sigma)^n$ such that $c_k = \psi^{\mathfrak{B}}[x_l/a_l]_{l\in 2}$, for all $k \in n$. Then, $\phi^{\mathfrak{B}}_i[x_l/a_l]_{l\in 2} \in D^{\mathcal{B}} \not\ni \phi^{\mathfrak{B}}_{1-i}[x_l/a_l]_{l\in 2}$, where, for all $m \in 2$, $\phi_m \triangleq (\varphi[x_0/x_m; x_{k+1}/\psi_k]_{k\in n}) \in \mathrm{Fm}^2_\Sigma$. Also, $(\phi_0[x_1/x_0]) = (\varphi[x_{k+1}/(\psi_k[x_1/x_0])]_{k\in n}) = (\phi_1[x_1/x_0])$, in which case $(\phi_i \vdash \phi_{1-i}) \in \varepsilon^2_\Sigma$, and so $\mathcal{B} \not\models (\bigwedge \varepsilon^2_\Sigma)[x_l/a_l]_{l\in 2}$. Hence, $\mathcal{A} \not\models (\bigwedge \varepsilon^2_\Sigma)[x_l/a_l]_{l\in 2}$, for $\bigwedge \varepsilon^2_\Sigma$ is quantifier-free. Thus, (viii) holds, as required. □

In this way, combining Remarks 3.7 and 3.8 with Theorem 3.9(iii)⇒(viii), we eventually get:

Corollary 3.10. *Any class M of $\sqsupset$-implicative Σ-matrices with (not necessarily same) equality determinant has an axiomatic canonical one, in which case this is that for the class of Σ-matrices $\mathbf{I}(\mathbf{S}(\mathbf{P}(\mathsf{M})))$ (being the least one including M and closed under $\mathbf{I}$, $\mathbf{S}$ and $\mathbf{P}$), and so every member of it is simple.*

3.4. **Semantics of structural completions of many-valued logics.** Let M be a class of Σ-matrices, C the logic of M, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and $\alpha \in \wp_{\omega[\setminus 1]}(\omega)$ [unless Σ contains a nullary connective]. Then, for any $\mathcal{A} \in \mathsf{M}$ and any $h \in \hom(\mathrm{Fm}^\alpha_\Sigma, \mathfrak{A})$, we have $\theta^\alpha_{\mathsf{K}} \subseteq (\ker h) = h^{-1}[\Delta_A] \subseteq h^{-1}[\theta^{\mathcal{A}}]$. On the other hand, $\mathcal{D} \triangleq \langle \mathfrak{Fm}^\alpha_\Sigma, \mathrm{Fm}^\alpha_\Sigma \cap C(\varnothing)\rangle \in \mathrm{Mod}(C)$, in view of the structurality of C, while, by definition of Cn_{M}, $D^{\mathcal{D}} = (\mathrm{Fm}^\alpha_\Sigma \cap \bigcap\{h^{-1}[D^{\mathcal{A}}] \mid \mathcal{A} \in \mathsf{M}, h \in \hom(\mathrm{Fm}^\alpha_\Sigma, \mathfrak{A})\})$, in which case $(\mathrm{Eq}^\alpha_\Sigma \cap \bigcap\{h^{-1}[\theta^{\mathcal{A}}] \mid \mathcal{A} \in \mathsf{M}, h \in \hom(\mathrm{Fm}^\alpha_\Sigma, \mathfrak{A})\}) \subseteq \theta^{\mathcal{D}}$, and so $\theta^\alpha_{\mathsf{K}} \subseteq \theta^{\mathcal{D}}$. Thus, $\theta^\alpha_{\mathsf{K}} \in \mathrm{Con}(\mathcal{D})$, in which case $\mathcal{F}^\alpha_{\mathsf{M}} \triangleq (\mathcal{D}/\theta^\alpha_{\mathsf{K}}) \in \mathrm{Mod}(C)$, while $\mathfrak{F}^\alpha_{\mathsf{M}} = \mathfrak{F}^\alpha_{\mathsf{K}}$.

Theorem 3.11. *Let Σ be a signature [with(out) nullary connectives], M a [finite (non-empty)] class of [finite] Σ-matrices, C the logic of M, [$f \in \prod_{\mathcal{A}\in\mathsf{M}} \wp_{\omega(\setminus 1)}(A)$] $\alpha \triangleq (\omega[\cap \bigcup_{\mathcal{A}\in\mathsf{M}} |f(\mathcal{A})|])$ and $\mathcal{B}$ a submatrix of $\mathcal{F}^{\alpha}_{\mathsf{M}}$. Suppose every $\mathcal{A} \in \mathsf{M}$ is a surjective homomorphic image of $\mathcal{B}$, unless $\mathcal{B} = \mathcal{F}^{\alpha}_{\mathsf{M}}$, [and generated by $f(\mathcal{A})$]. Then, the structural completion of C is defined by $\mathcal{B}$.*

Proof. Then, the logic C' of $\mathcal{F}^{\omega[/\alpha]}_{\mathsf{M}}$ is defined by $\mathcal{D}_{\omega[/\alpha]} \triangleq \langle \mathfrak{Fm}^{\omega[/\alpha]}_{\Sigma}, \mathrm{Fm}^{\omega[/\alpha]}_{\Sigma} \cap C(\varnothing)\rangle \in \mathrm{Mod}(C)$, in view of the structurality of C, in which case it is an extension of C, and so $C(\varnothing) \subseteq C'(\varnothing)$. For proving the converse inclusion, consider the following complementary cases:

- $\alpha = \omega$.
  Then, applying the diagonal Σ-substitution, we get $C'(\varnothing) \subseteq D^{\mathcal{D}_{\omega}} = C(\varnothing)$.
- $\alpha \neq \omega$.
  Consider any $\mathcal{A} \in \mathsf{M}$, in which case it is generated by $f(\mathcal{A})$ of cardinality $\leqslant \alpha$, and so there is some surjective $h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$. Then, $D^{\mathcal{D}_{\alpha}} = (\mathrm{Fm}^{\alpha}_{\Sigma} \cap C(\varnothing)) \subseteq h^{-1}[D^{\mathcal{A}}]$, in which case $h \in \hom^{\mathrm{S}}(\mathcal{D}_{\alpha}, \mathcal{A})$, and so $C'(\varnothing) \subseteq C(\varnothing)$.

Next, $\mathcal{D}_{\omega}$ is a model of any axiomatically-equivalent extension C'' of C' , in view of its structurality [and so is its submatrix $\mathcal{D}_{\alpha}$], in which case C' is the structural completion of C. Finally, $\mathcal{B}$ is a model of C'. Conversely, if $\mathcal{B} = \{\neq\}\mathcal{F}^{\omega[/\alpha]}_{\mathsf{M}}$, then {each $\mathcal{A} \in \mathsf{M}$ is a surjective homomorphic image of $\mathcal{B}$, in which case} $\mathrm{Cn}_{\mathcal{B}}(\varnothing)\{= C(\varnothing)\} = C'(\varnothing)$, and so C', being structurally complete, is defined by $\mathcal{B}$. □

The []-optional case of this theorem provides an effective procedure of finding finite matrix semantics of any finitely-valued logic, practical applications of which are demonstrated in Subsection 8.2 below.

4. Super-classical matrices versus 3VLPSN

Set $\neg^{\wr}_{\diamond} x_0 \triangleq (\wr x_0 \diamond \wr(x_0 \diamond \wr x_0))$ and $(x_0 \sqsupset^{\wr}_{\overline{\wedge},\underline{\vee}} x_1) \triangleq (x_0 \sqsupset^{\neg^{\wr}_{\overline{\wedge}}}_{\underline{\vee}} x_1)$. From now on, unless otherwise specified, it is supposed that $\wr = (\sim | \neg^{[\wr]}_{[\diamond]})$, while $\diamond = (\overline{\wedge} | \underline{\vee} | \sqsupset^{[\wr]}_{[(\overline{\wedge},)\underline{\vee}]} | \wedge | \vee_{[\diamond]} | \supset)$, depending upon the context.

A Σ-matrix $\mathcal{A}$ is said to be *(canonically) $\sim$-super-classical*, whenever $A = (3 \div 2)$, $\sim^{\mathfrak{A}} \frac{1}{2} \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$ and $\sim^{\mathfrak{A}} k = (1 - k)$, for all $k \in 2$, in which case it is both false-singular and $\sim$-paraconsistent, while the *Inverse Double Negation* rule:

$$\sim\sim x_0 \vdash x_0 \tag{4.1}$$

is true in $\mathcal{A}$, whereas 2 forms a subalgebra of $\mathfrak{A}{\restriction}\{\sim\}$, $(\mathcal{A}{\restriction}\{\sim\}){\restriction}2$ being canonically $\sim$-classical, and so we have the routine part (viz., (iii)⇒(iv)⇒(ii)⇒(i)) of the following preliminary semantic marking the framework of the present study:

Theorem 4.1. *Let C be a Σ-logic. Then, the following are equivalent:*

(i) *C is three-valued, $\sim$-paraconsistent, while either $\sim$ is a subclassical negation for C or both* (4.1) *is satisfied in C and* (2.11) *with $(m + n) = 1$ holds;*
(ii) *C is three-valued, $\sim$-paraconsistent, while both $\sim$ is a subclassical negation for C and* (4.1) *is satisfied in C;*
(iii) *C is three-valued, while any three-valued Σ-matrix defining C is isomorphic to a $\sim$-super-classical Σ-matrix;*
(iv) *C is defined by a $\sim$-super-classical Σ-matrix.*

Proof. Assume (i) holds. Let $\mathcal{B}$ be any three-valued Σ-matrix defining C. Define an $e : (2 \cup \{\frac{1}{2}\}) \to B$ as follows. In that case, $\mathcal{B}$ is $\sim$-paraconsistent, so there are some $e(\frac{1}{2}) \in D^{\mathcal{B}}$ such that $\sim^{\mathfrak{B}} e(\frac{1}{2}) \in D^{\mathcal{B}}$ and some $e(0) \in (B \setminus D^{\mathcal{B}})$, in which case $e(0) \neq e(\frac{1}{2})$. Next, by (2.11) with $m = 1$ and $n = 0$, there is some $e(1) \in D^{\mathcal{B}}$ such that $\sim^{\mathfrak{B}} e(1) \notin D^{\mathcal{B}}$, in which case $e(0) \neq e(1) \neq e(\frac{1}{2})$. In this way, e is injective, and so bijective, for $|B| = 3$. Hence, it is an isomorphism from $\mathcal{A} \triangleq \langle e^{-1}[\mathfrak{B}], \{\frac{1}{2}, 1\}\rangle$ onto $\mathcal{B}$. Therefore, C is defined by $\mathcal{A}$. Furthermore, $\sim^{\mathfrak{A}} \frac{1}{2} \in D^{\mathcal{A}}$, while $\sim^{\mathfrak{A}} 1 \notin D^{\mathcal{A}}$, in which case $\sim^{\mathfrak{A}} 1 = 0$, and so, for proving that $\mathcal{A}$ is $\sim$-super-classical, in which case (iii) holds, it only remains to show that $\sim^{\mathfrak{A}} 0 = 1$. We do it by contradiction. For suppose $\sim^{\mathfrak{A}} 0 \neq 1$, in which case we have the following complementary cases:

- $\sim^{\mathfrak{A}} 0 = 0$.
  This contradicts to (2.11) with $m = 0$ and $n = 1$.
- $\sim^{\mathfrak{A}} 0 = \frac{1}{2}$,
  in which case (4.1) is not true in $\mathcal{A}$ under $[x_0/\frac{1}{2}]$, for $\sim^{\mathfrak{A}} \frac{1}{2} \in D^{\mathcal{A}}$, and so $\sim$ is a subclassical negation for C. Consider the following complementary subcases, for $\sim^{\mathfrak{A}} \frac{1}{2} \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$:
  - $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$.
    Then, $\sim^{\mathfrak{A}} \sim^{\mathfrak{A}} \sim^{\mathfrak{A}} a = \frac{1}{2} \in D^{\mathcal{A}}$, for each $a \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$. This contradicts to (2.11) with $m = 3$ and $n = 0$.
  - $\sim^{\mathfrak{A}} \frac{1}{2} = 1$.
    Then, $\sim^{\mathfrak{A}} \sim^{\mathfrak{A}} \sim^{\mathfrak{A}} 0 = 0$. This contradicts to (2.11) with $m = 0$ and $n = 3$.

Thus, anyway, we come to a contradiction, as required. □

From now on, unless otherwise specified, C is supposed to be the logic of an arbitrary but fixed $\sim$-super-classical Σ-matrix $\mathcal{A}$. (In view of Theorem 4.1, this exhaust all three-valued $\sim$-paraconsistent Σ-logics with subclassical negation $\sim$.) Then, $\mathcal{A}$ is said to be *classically-hereditary*, whenever 2 forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}\upharpoonright 2$ is a canonically $\sim$-classical model of C, and so this is $\sim$-subclassical.

In general, providing $(K|L)_{(6/5)|/5} \triangleq ((A \times 2) \setminus (\varnothing/\{\langle \frac{1}{2}, 0|1\rangle\}))$ forms a subalgebra of $\mathfrak{A}^2$, "that is"/"in which case" $2 = \pi_1[(K|L)_{(6/5)|/5}]$ forms a subalgebra of $\mathfrak{A}$, for $(\pi_1 \upharpoonright A^2) \in \hom(\mathfrak{A}^2, \mathfrak{A})$ is surjective, $(\mathcal{K}|\mathcal{L})_{(6/5)|/5} \triangleq (\mathcal{A}^2 \upharpoonright (K|L)_{(6/5)|/5}) \in \mathrm{Mod}(C^{\mathrm{NP}})$, the logic of which, being then an extension of C, is axiomatically-equivalent to this, for $(\pi_0 \upharpoonright A^2) \in \hom^{\mathrm{S}}((\mathcal{K}|\mathcal{L})_{(6/5)|/5}, \mathcal{A})$, respectively. Set $L_{2/3} \triangleq ((2^2 \setminus \Delta_2) \cup (\varnothing/\{\langle \frac{1}{2}, \frac{1}{2}\rangle\}))$, $L_4 \triangleq (A^2 \setminus (\Delta_2 \cup L_3))$, $K_2 \triangleq \{\langle 0, 1, \frac{1}{2}\rangle, \langle 0, 0, 0\rangle\}$ and $K_{4/3} \triangleq (K_{6/5} \setminus L_2) \subseteq A^2$, respectively. Then, by $\mathcal{K}'_{2|(5/3)}$ we denote the submatrix of $\mathcal{A}^{3|2}$ generated by $K_{2|(5/3)}$, respectively.

Next, a *[truth-]symmetric/idempotent formula for|of* $\mathcal{A}$ is any $\varphi \in \mathrm{Fm}^2_\Sigma$ such that, for all $\bar{a} \in (A[\cap D^{\mathcal{A}}])^{2/1}$, it holds that $\varphi^{\mathfrak{A}}(a_0, a_{1/0}) = (\varphi^{\mathfrak{A}}(a_1, a_0)/a_0)$, (Clearly, x_i, where $i \in 2$, is an idempotent formula for $\mathcal{A}$, not being a symmetric one, because $|A| = 3 \neq 1$.) Then, $\mathcal{A}$ is said to be *[truth-]symmetric*, whenever it has a [truth-]symmetric formula.

Further, $\mathcal{A}$ is said to be *classically-valued*, provided, for each $\varsigma \in \Sigma$, $(\mathrm{img}\, \varsigma^{\mathfrak{A}}) \subseteq 2$, in which case it is classically-hereditary, while $(K|L)_5$ forms a subalgebra of $\mathfrak{A}^2$, whereas x_0 and x_1 are the only [truth-]idempotent formulas for $\mathcal{A}$, because $2 \not\ni \frac{1}{2}[\in D^{\mathcal{A}}]$, and so $\mathfrak{A}$ is not a $\diamond$-semi-lattice (in particular, the scopes of Subsubsections 8.1 and 8.2 are disjoint).

Remark 4.2. Suppose $\mathcal{A}$ is both classically-valued and $\diamond$-conjunctive/"-disjunctive (in particular, $\sqsupset$-imlicative, while $\diamond = \vee_{\sqsupset}$)", in which case $\diamond$ is a symmetric formula for $\mathcal{A}$. Then, as $1 \in D^{\mathcal{A}} \not\ni 0$, we have $(a \diamond^{\mathfrak{A}} a) = \chi^{\mathcal{A}}(a)$, for all $a \in A$, in which case, since $\sim^{\mathfrak{A}} i = (1 - i)$, for all $i \in 2$, $\mathcal{A}$ is $\neg$-negative, where $\neg x_0 \triangleq \sim(x_0 \diamond x_0)$, and so both $\overline{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive, where $\overline{\wedge} \triangleq (\diamond/\diamond^{\neg})$ and $\underline{\vee} \triangleq (\diamond^{\neg}/\diamond)$, as well as $\sqsupset_{\underline{\vee}}$-implicative. □

Remark 4.3 (cf. Example 2 of [17]). $\sim$ is a canonical unitary equality determinant for $\mathcal{A}$. In particular, in case $\mathcal{A}$ is [both $\overline{\wedge}$-conjunctive and] $\sqsupset$-implicative, by Remark 3.7, $\varepsilon^{\sim+}_{\sqsupset}$ is an axiomatic canonical equality determinant for $\mathcal{A}$ [and so is $\varepsilon^{\sim}_{\sqsupset,\overline{\wedge}} \triangleq (\overline{\wedge}\langle\overline{\wedge}\langle\sim^{i}x_{j} \sqsupset \sim^{i}x_{1-j}\rangle_{j\in 2}\rangle_{i\in 2})$]. $\square$

Further, a *((2[+1])-ary [$\frac{1}{2}$-relative] {classical}) semi-conjunction for/of* a canonically $\sim$-(super-)classical Σ-matrix $\mathcal{B}$ is an arbitrary $\varphi \in \mathrm{Fm}^{2([+1])}_{\Sigma}$ such that both $\varphi^{\mathfrak{B}}(0,1([,\frac{1}{2}])) = 0$ and $\varphi^{\mathfrak{B}}(1,0([,\frac{1}{2}])) \in \{0([,\frac{1}{2}])\}$. (Clearly, any binary semi-conjunction for $\mathcal{B}$ is a ternary $\frac{1}{2}$-relative one.)

Lemma 4.4 (1st Key Lemma). *Let $\mathcal{B}$ be a $\sim$-paraconsistent model of C. Suppose either $\mathcal{A}$ has a ternary $\frac{1}{2}$-relative semi-conjunction or $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$ or*

$$x_0 \vdash \sim x_0 \tag{4.2}$$

is not true in $\mathcal{B}$. Then, $\mathcal{A}$ is embeddable into a strict surjective homomorphic image of a $\sim$-paraconsistent submatrix of $\mathcal{B}$.

Proof. Then, by Lemma 3.6 and Remark 4.3, $\mathcal{A}$ is simple. Moreover, [in case (4.2) is not true in $\mathcal{B}$] there are some $a, b[, c] \in B$ such that $D^{\mathcal{B}} \supseteq \{\sim^{\mathfrak{B}}a[, c]\}$ is disjoint with $\{b[, \sim^{\mathfrak{B}}c]\}$. Therefore, the submatrix $\mathcal{D}$ of $\mathcal{B}$ generated by $\{a, b[, c]\}$ is a finitely-generated $\sim$-paraconsistent model of C [in which (4.2) is not true under $[x_0/c]$]. Hence, by Lemma 2.2, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_{*}(\mathcal{A})^{I}$, some subdirect product $\mathcal{E}$ of it, some strict surjective homomorphic image $\mathcal{F}$ of $\mathcal{D}$ and some $h \in \hom_{\mathrm{S}}(\mathcal{E}, \mathcal{F})$, in which case $\mathcal{E}$ is $\sim$-paraconsistent, and so consistent (in particular, $I \neq \varnothing$) [while (4.2) is not true in $\mathcal{E}$]. Given any $a' \in A$ and any $J \subseteq I$, set $(J : a') \triangleq (J \times \{a'\}) \in A^{J}$. Likewise, given any $\bar{a} \in A^2$ and any $J \subseteq I$, set $(a_0 :_J a_1) \triangleq ((J : a_0) \cup ((I \setminus J) : a_1)) \in A^{I}$. Then, there are some $d \in (E \setminus D^{\mathcal{E}})$ and some $e[, f] \in D^{\mathcal{E}}$ such that $\sim^{\mathfrak{E}}e \in D^{\mathcal{E}}[\not\ni \sim^{\mathfrak{E}}f]$, in which case $e = (I : \frac{1}{2})$ and $J \triangleq \{i \in I \mid \pi_i(d) = 0\} \neq \varnothing[\neq K \triangleq \{i \in I \mid \pi_i(f) = 1\}]$. Consider the following complementary cases:

- $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$,
  in which case $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$. We are going to prove that there is some non-empty $L \subseteq I$ such that $(0 :_L \frac{1}{2}) \in E$. For consider the following exhaustive subcases:
  - $\mathcal{A}$ has a ternary $\frac{1}{2}$-relative semi-conjunction φ.
    Let $g \triangleq \varphi^{\mathfrak{E}}(d, \sim^{\mathfrak{E}}d, e)$. Consider the following exhaustive subsubcases:
    * $\varphi^{\mathfrak{A}}(1, 0, \frac{1}{2}) = 0$.
      Let $L \triangleq \{i \in I \mid \pi_i(d) \neq \frac{1}{2}\} \supseteq J$. Then, $E \ni g = (0 :_L \frac{1}{2})$.
    * $\varphi^{\mathfrak{A}}(1, 0, \frac{1}{2}) = \frac{1}{2}$.
      Let $L \triangleq J$. Then, $E \ni g = (0 :_L \frac{1}{2})$.
  - (4.2) is not true in $\mathcal{B}$.
    Let $L \triangleq K$. Then, $f \in D^{\mathcal{E}} \subseteq \{\frac{1}{2}, 1\}^{I}$, in which case $E \ni f = (1 :_L \frac{1}{2})$, and so $E \ni \sim^{\mathfrak{E}}f = (0 :_L \frac{1}{2})$.

  In this way, $(0 :_L \frac{1}{2}) \in E \ni e = (\frac{1}{2} :_L \frac{1}{2})$, in which case $E \ni \sim^{\mathfrak{E}}(0 :_L \frac{1}{2}) = (1 :_L \frac{1}{2})$, and so, as $L \neq \varnothing$, while $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, $h' \triangleq \{\langle x, (x :_L \frac{1}{2})\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{E}$.
- $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$,
  in which case there is some $\varphi \in \mathrm{Fm}^{1}_{\Sigma}$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2}) \in 2$, and so $A = \{\frac{1}{2}, \varphi^{\mathfrak{A}}(\frac{1}{2}), \sim^{\mathfrak{A}}\varphi^{\mathfrak{A}}(\frac{1}{2})\}$. Hence, $\{I : x \mid x \in A\} = \{e, \varphi^{\mathfrak{E}}(e), \sim^{\mathfrak{E}}\varphi^{\mathfrak{E}}(e)\} \subseteq E$. Therefore, as $I \neq \varnothing$, $h' \triangleq \{\langle x, I : x\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{E}$.

Thus, $(h \circ h') \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{F})$ is injective, for $\mathcal{A}$ is simple, as required. $\square$

Theorem 4.5. *Any $\sim$-super-classical $\mathcal{B} \in \mathrm{Mod}(C)$ is equal to $\mathcal{A}$.*

Proof. By Lemma 3.6 and Remark 4.3, $\mathcal{B}$ is simple. Moreover, it has no proper $\sim$-paraconsistent submatrix. And what is more, by Theorem 4.1(iv)$\Rightarrow$(i) and (2.11) with $n = 0$ and $m = 1$, (4.2) is not true in $\mathcal{B}$. Hence, by Lemma 4.4, $\mathcal{A}$ is embeddable into $\mathcal{B}$, and so isomorphic to this, for it has no proper $\sim$-paraconsistent submatrix. Take any bijective $e \in \mathrm{hom}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, in which case it is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$, and so $(\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}) \Leftrightarrow (\mathfrak{A} \models (\exists x_0(\sim x_0 \approx x_0))) \Leftrightarrow (\mathfrak{B} \models (\exists x_0(\sim x_0 \approx x_0))) \Leftrightarrow (\sim^{\mathfrak{B}} \frac{1}{2} = \frac{1}{2})$. In this way, $\sim^{\mathfrak{A}} = \sim^{\mathfrak{B}}$, in which case $e \in \mathrm{hom}_{\mathrm{S}}(\mathcal{D}, \mathcal{D})$, where $\mathcal{D} \triangleq (\mathcal{A} \restriction \{\sim\}) = (\mathcal{B} \restriction \{\sim\})$ is $\sim$-super-classical, and so, for all $a \in D$ and all $k \in 2$, we have $((\sim^{\mathfrak{D}})^k a \in D^{\mathcal{D}}) \Leftrightarrow ((\sim^{\mathfrak{D}})^k e(a) \in D^{\mathcal{D}})$. Therefore, by Remark 4.3, we get $e(a) = a$, in which case e is diagonal, and so $\mathcal{B} = \mathcal{A}$. $\square$

In view of Theorem 4.5, $\mathcal{A}$ is determined uniquely by C, and so is referred to as *characteristic for/of C*.

Corollary 4.6. *Let $\Sigma' \supseteq \Sigma$ be a signature and C' a three-valued Σ'-expansion of C. Then, C' is defined by a unique Σ'-expansion of $\mathcal{A}$.*

Proof. In that case, $\sim$ is a subclassical negation for C', being, in its turn, $\sim$-paraconsistent. Hence, by Theorem 4.1(i)$\Rightarrow$(iv), C' is defined by a $\sim$-super-classical Σ'-matrix $\mathcal{A}'$, in which case C is defined by the $\sim$-super-classical Σ-matrix $\mathcal{A}' \restriction \Sigma$, and so, by Theorem 4.5, this is equal to $\mathcal{A}$. Finally, as any Σ'-expansion of $\mathcal{A}$ is $\sim$-super-classical, Theorem 4.5 completes the argument. $\square$

4.1. **Examples.** Let $\Sigma^{\{\supset\}}_{\sim(,+[,01])} \triangleq ((\Sigma_{+[,01]} \cup)\{\sim\}\{\cup\{\supset\}\})$ {where $\supset$ is binary}. Note that all the *particular* $\sim$-super-classical Σ-matrices discussed here are classically-hereditary and conjunctive, in which case their logics are $\sim$-classical and conjunctive, and so are covered by the reference [Pyn 95b] of [13].

4.1.1. *The logic of paradox and its expansions.* Let $\Sigma \triangleq \Sigma_{\sim,+[,01]}$, $\sim^{\mathfrak{A}} \frac{1}{2} \triangleq \frac{1}{2}$ and $(\mathfrak{A} \restriction \Sigma_{+[,01]}) \triangleq \mathfrak{D}_{3[,01]}$, in which case $\mathcal{A}$ is both $\wedge$-conjunctive and $\vee$-disjunctive. Then, C is [the bounded expansion of] the *logic of paradox* $LP_{[01]}$ (cf. [11, 13]).
4.1.1.1. Sugihara odd-valued logics and their bounded expansions. Let $n \in (\omega \setminus 2)$, $\Sigma \triangleq \Sigma^{\supset}_{\sim,+[,01]}$ and $\mathcal{S}_{n[,01]}$ the Σ-matrix with $(\mathfrak{S}_{n[,01]} \restriction \Sigma_{+[,01]}) \triangleq \mathfrak{D}_{n[,01]}$, $D^{\mathcal{S}_{n[,01]}} \triangleq \{a \in (n \div (n-1)) \mid 1 \leqslant (2 \cdot a)\}$, $\sim^{\mathfrak{S}_{n[,01]}} a \triangleq (1 - a)$, for all $a \in (n \div (n-1))$, and

$$(a \supset^{\mathfrak{S}_{n[,01]}} b) \triangleq \begin{cases} \max(1-a, b) & \text{if } a \leqslant b, \\ \min(1-a, b) & \text{otherwise}, \end{cases}$$

for all $a, b \in (n \div (n-1))$, in which case $\mathcal{S}_{n[,01]}$ is both $\wedge$-conjunctive and $\veebar$-disjunctive, while $\mathcal{S}_{2[,01]}$ is a canonically $\sim$-classical submatrix of $\mathcal{S}_{n[,01]}$. Then, the logic $\mathbb{S}_{n[,01]}$ of $\mathcal{S}_{n[,01]}$ is [the bounded expansion of] Sugihara n-valued logic [23] and is $\sim$-subclassical. Clearly, $\mathbb{S}_{3[,01]}$ is a three-valued expansion of $LP_{[01]}$ (in particular, it is $\sim$-paraconsistent). And what is more, in case n is odd, $\mathcal{S}_{3[,01]}$ is a submatrix of $\mathcal{S}_{n[,01]}$, in which case $\mathbb{S}_{n[,01]}$ is a sublogic of $\mathbb{S}_{3[,01]}$, and so is $\sim$-paraconsistent, while $h_n \triangleq (\{\langle \frac{1}{2}, \frac{1}{2} \rangle\} \cup (\chi^{\mathcal{S}_{n[,01]}} \restriction ((n \div (n-1)) \setminus \{\frac{1}{2}\})) \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{S}_{n[,01]} \restriction \Sigma_{\sim,+[,01]}, \mathcal{S}_{3[,01]} \restriction \Sigma_{\sim,+[,01]})$, in which case $\mathbb{S}_{n[,01]}$ is an expansion of $LP_{[01]}$, whereas $\mathcal{S}_{3[,01]}$ is $\sqsupset$-implicative, where $(x_0 \sqsupset x_1) \triangleq ((x_0 \supset x_1) \vee x_1)$, in which case $(a \sqsupset^{\mathfrak{S}_{3[,01]}} = (\max(1 - \chi^{\mathcal{S}_{3[,01]}}(a), b)$, for all $a, b \in (3 \div 2)$ (conversely, the Σ-identity $(x_0 \supset x_1) \approx (\wedge \langle \sim^i x_i \sqsupset \sim^i x_{1-i} \rangle_{i \in 2})$ is true in $\mathfrak{S}_{3[,01]}$ — this is why $\mathbb{S}_{3[,01]}$, up to term-wise definitional equivalence, was actually studied in Subsection 5.3 of [19]), in which case the Σ-axiom $x_0 \vee (x_0 \sqsupset x_1)$, not being true in $\mathcal{S}_{n[,01]}$ under $[x_0/\frac{1}{n-1}, x_1/0]$, unless $n = 3$, is true in $\mathcal{S}_{3[,01]}$, for this is $\vee$-disjunctive, and so $\mathbb{S}_{n[,01]}$ is a proper sublogic of $\mathbb{S}_{3[,01]}$ (in particular, it is not maximally $\sim$-paraconsistent), unless $n = 3$. Otherwise, $\mathcal{S}_{n[,01]}$ is non-$\sim$-paraconsistent, and so is $\mathbb{S}_{n[,01]}$, in which case this is not a sublogic/expansion of $\mathbb{S}_{3[,01]}/LP_{[01]}$, respectively.

4.1.2. *Hałkowska-Zajac's logic.* Let $\Sigma \triangleq \Sigma_{\sim,+}$, $\sim^{\mathfrak{A}}\frac{1}{2} \triangleq \frac{1}{2}$ and $\mathfrak{A}$ the $(\wedge,\vee)$-lattice with zero $\frac{1}{2}$ and unit 1. Then, C is HZ [4].

Note that $\mathcal{A}$ is neither $\wedge$-conjunctive nor $\vee$-disjunctive. Nevertheless, the identity $\sim\sim x_0 \approx x_0$ is satisfied in $\mathfrak{A}$, in which case $\mathfrak{A}$ is a $(\vee^{\sim},\wedge^{\sim})$-lattice with zero $\sim^{\mathfrak{A}}1 = 0$ and unit $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$, and so $\mathcal{A}$ is both $\vee^{\sim}$-conjunctive and $\wedge^{\sim}$-disjunctive. And what is more, it is $\sqsupset$-implicative, where $(x_0 \sqsupset x_1) \triangleq ((\sim x_0 \wedge \sim x_1) \vee x_1)$.

On the other hand, since the identity $\sim\sim x_0 \approx x_0$ is satisfied in $\mathfrak{A}$, so are both $(x_0(\vee|\wedge)x_1) \approx (x_0((\vee|\wedge)^{\sim})^{\sim}x_1)$, in which case $(\vee|\wedge)^{\mathfrak{A}}$ become secondary operations of $\mathfrak{A}$, while taking $((\vee|\wedge)^{\sim})^{\mathfrak{A}}$ as primary ones. This well justifies the modification of HZ considered below.

4.1.2.1. A non-idempotent counterpart. Let $\Sigma \triangleq \Sigma_{\sim,+}$, $\sim^{\mathfrak{A}}\frac{1}{2} \triangleq 1$ and $\mathfrak{A}$ the $(\wedge,\vee)$-lattice with zero 0 and unit $\frac{1}{2}$. Then, C is actually a non-idempotent counterpart $NIHZ$ of HZ [4]. Clearly, $\mathcal{A}$ is both $\wedge$-conjunctive and $\veebar$-disjunctive. And what is more, $\mathcal{A}$ is $\neg^{\sim}_{\wedge}$-negative, and so $\sqsupset^{\sim}_{\wedge,\vee}$-implicative.

4.1.3. *Sette's logic.* Let $\Sigma = \Sigma^{\supset}_{\sim}$ and $\mathcal{A}$ both classically-valued and $\supset$-implicative (in which case it is $\vee_{\supset}$-disjunctive, and so conjunctive; cf. Remark 4.2). Then, C is P^1 [22].

4.1.4. *Paraconsistent counterparts of Gödel three-valued logic and its implication-less fragment.*
4.1.4.1. The implication-less fragment. Let $\Sigma \triangleq \Sigma_{\sim,+,01}$, $\sim^{\mathfrak{A}}\frac{1}{2} \triangleq 1$, in which case $\sim^{\mathfrak{A}}$ is dual pseudo-complement, and $(\mathfrak{A}{\restriction}\Sigma_{+,01}) \triangleq \mathfrak{D}_{3,01}$, in which case $\mathcal{A}$ is both $\wedge$-conjunctive and $\vee$-disjunctive. Then, C is a paraconsistent counterpart $PG3^*$ of the implication-less fragment of Gödel three-valued logic [3].
4.1.4.2. The full version. Let $\Sigma \triangleq \Sigma^{\supset}_{\sim,+,01}$ and C is the three-valued Σ-expansion of $PG3^*$ given by $(a \supset^{\mathfrak{A}} b) \triangleq \min\{c \in A \mid b \leqslant \max(a,c)\}$, for all $a, b \in A$, in which case $\supset^{\mathfrak{A}}$ is dual relative pseudo-complement, while $\mathcal{A}$ is not $\supset$-implicative (like Gödel three-valued logic [3] not satisfying (2.8)), because (neither) (2.6) (nor (2.7) nor (2.8)) is true in it under $[x_0/0(,x_1/0)]$, though (2.5) is true in it, as $(a \supset^{\mathfrak{A}} 0) = 0$, for all $a \in A$, and so, by Lemma 3.3, C is not $\supset$-implicative, though it satisfies (2.5). Then, C is a paraconsistent counterpart $PG3$ of Gödel three-valued logic [3].

5. Paraconsistent extensions

Theorem 5.1. *The following are equivalent [provided C is $\sim$-subclassical]:*

(i) *C has no proper $\sim$-paraconsistent [$\sim$-subclassical] extension;*
(ii) *C has no proper $\sim$-paraconsistent non-$\sim$-subclassical extension;*
(iii) *either $\mathcal{A}$ has a ternary $\frac{1}{2}$-relative semi-conjunction or $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$ (in particular, $\sim^{\mathfrak{A}}\frac{1}{2} \neq \frac{1}{2}$);*
(iv) *L_3 does not form a subalgebra of $\mathfrak{A}^2$;*
(v) *$\mathcal{A}$ has no truth-singular $\sim$-paraconsistent subdirect square;*
(vi) *$\mathcal{A}^2$ has no truth-singular $\sim$-paraconsistent submatrix;*
(vii) *C has no truth-singular $\sim$-paraconsistent model;*
(viii) *$\mathcal{A}_{\frac{1}{2}} \triangleq \langle\mathfrak{A}, \{\frac{1}{2}\}\rangle$ is not a $\sim$-paraconsistent model of C;*
(ix) *C has no truth-singular $\sim$-paraconsistent model with underlying algebra $\mathfrak{A}$.*

In particular, C has a $\sim$-paraconsistent proper extension iff it has a [non-]non-$\sim$-subclassical one, and if any three-valued expansion of C does so.

Proof. First, assume (iii) holds. Consider any $\sim$-paraconsistent extension C' of C, in which case $x_1 \notin T \triangleq C'(\{x_0, \sim x_0\}) \supseteq \{x_0, \sim x_0\}$, and so, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a $\sim$-paraconsistent model of C' (in particular, of C). Hence, by Lemma 4.4, $\mathcal{A}$ is a model of C', in which case $C' = C$, and so both (i) and (ii) hold.

Next, assume L_3 forms a subalgebra of $\mathfrak{A}^2$. Then, $\mathcal{B} \triangleq (\mathcal{A}^2 \restriction L_3) \in \mathrm{Mod}(C)$ is a subdirect square of $\mathcal{A}$, because $\pi_i[L_3] = A$, for each $i \in 2$. Moreover, L_2 is disjoint with $D^{\mathcal{B}} \ni \langle \frac{1}{2}, \frac{1}{2}\rangle$, for $0 \notin D^{\mathcal{A}} \ni \frac{1}{2}$, in which case we have $D^{\mathcal{B}} = \{\langle \frac{1}{2}, \frac{1}{2}\rangle\} = (L_3 \cap \Delta_A)$, and so $\mathcal{B}$ is both truth-singular and, being consistent, for $L_3 \supseteq L_2 \neq \varnothing$, $\sim$-paraconsistent, for $L_3 \ni \sim^{\mathfrak{A}^2}\langle \frac{1}{2}, \frac{1}{2}\rangle = \langle \sim^{\mathfrak{A}}\frac{1}{2}, \sim^{\mathfrak{A}}\frac{1}{2}\rangle \in \Delta_A$. Moreover, $(\pi_0 \restriction L_3) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}, \mathcal{A}_{\frac{1}{2}})$. Hence, $\mathcal{A}_{\frac{1}{2}} \in \mathrm{Mod}(C)$ is $\sim$-paraconsistent. Thus, (v/viii)$\Rightarrow$(iv) holds, while (v/viii/ix) is a particular case of (vi/ix/vii), respectively, whereas (vii)$\Rightarrow$(vi) is immediate.

Now, let $\mathcal{B} \in \mathrm{Mod}(C)$ be both $\sim$-paraconsistent and truth-singular, in which case (4.2) is true in $\mathcal{B}$, and so is its logical consequence

$$\{x_0, x_1, \sim x_1\} \vdash \sim x_0, \tag{5.1}$$

not being true in $\mathcal{A}$ under $[x_0/1, x_1/\frac{1}{2}]$ [but, being a logical consequence of (2.9)$[x_0/x_1, x_1/\sim x_0]$, true in any $\sim$-classical model $\mathcal{C}'$ of C]. Thus, the logic of $\{\mathcal{B}[, \mathcal{C}']\}$ is a proper $\sim$-paraconsistent [$\sim$-subclassical] extension of C, so (i)$\Rightarrow$(vii) holds. And what is more, (4.2), being true in $\mathcal{B}$, is not true in any $\sim$-[super-]classical Σ-matrix [in particular, in $\mathcal{A}$], in view of [Theorem 4.1 and] (2.11) with $n = 0$ and $m = 1$. Thus, the logic of $\mathcal{B}$ is a proper $\sim$-paraconsistent non-$\sim$-subclassical extension of C, so (ii)$\Rightarrow$(vii) holds.

Finally, assume $\mathcal{A}$ has no ternary $\frac{1}{2}$-relative semi-conjunction and $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}^2$ generated by L_3. If $\langle 0, 0\rangle$ was in B, then there would be some $\varphi \in \mathrm{Fm}^3_{\Sigma}$ such that $\varphi^{\mathfrak{A}}(0, 1, \frac{1}{2}) = 0 = \varphi^{\mathfrak{A}}(1, 0, \frac{1}{2})$, in which case it would be a ternary $\frac{1}{2}$-relative semi-conjunction for $\mathcal{A}$. Likewise, if either $\langle \frac{1}{2}, 0\rangle$ or $\langle 0, \frac{1}{2}\rangle$ was in B, then there would be some $\varphi \in \mathrm{Fm}^3_{\Sigma}$ such that $\varphi^{\mathfrak{A}}(0, 1, \frac{1}{2}) = 0$ and $\varphi^{\mathfrak{A}}(1, 0, \frac{1}{2}) = \frac{1}{2}$, in which case it would be a ternary $\frac{1}{2}$-relative semi-conjunction for $\mathcal{A}$. Therefore, as $\sim^{\mathfrak{A}}1 = 0$ and $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$, we conclude that $(\{\langle 0, \frac{1}{2}\rangle, \langle 1, \frac{1}{2}\rangle, \langle \frac{1}{2}, 1\rangle, \langle \frac{1}{2}, 0\rangle, \langle 0, 0\rangle, \langle 1, 1\rangle\} \cap B) = \varnothing$. Thus, $B = L_3$ forms a subalgebra of $\mathfrak{A}^2$. In this way, (iv)$\Rightarrow$(iii) holds.

After all, Corollary 4.6 completes the argument, for any expansion of $\mathcal{A}$ inherits ternary $\frac{1}{2}$-relative semi-conjunctions (if any). □

Theorem 5.1(i)$\Leftrightarrow$(iii[iv]) is especially useful for [effective dis]proving the maximal $\sim$-paraconsistency of C, as we show below [cf. Example 5.10]. And what is more, since, $\mathcal{A}$ has no proper $\sim$-paraconsistent submatrix, by Corollary 2.4 and Theorem 4.1, we immediately have the following "axiomatic" version of Theorem 5.1 subsuming [22]:

Proposition 5.2. *Any $\sim$-paraconsistent three-valued Σ-logic with subclassical negation $\sim$ is axiomatically maximally $\sim$-paraconsistent.*

Remark 5.3. Suppose $\mathcal{A}$ is weakly $\bar{\wedge}$-conjunctive. Then, $(x_0 \mathbin{\bar{\wedge}} x_1)$ is a binary semi-conjunction for $\mathcal{A}$. □

By Theorems 4.1, 5.1(iii)$\Rightarrow$(i) and Remark 5.3, we first have:

Corollary 5.4 (cf. the reference [Pyn 95b] of [13])**.** *C is maximally $\sim$-paraconsistent, whenever it is weakly conjunctive. Any weakly conjunctive three-valued $\sim$-paraconsistent Σ-logic with subclassical negation $\sim$ is maximally $\sim$-paraconsistent.*

The principal advance of this universal *maximal paraconsistency* result with regard to its particular case obtained in the reference [Pyn 95b] of [13] but for merely $\sim$-subclassical logics, subsuming particular results first obtained *ad hoc* for LP (being $\wedge$-conjunctive) in [13], HZ in [16] and $\mathbb{S}_3$ in [19], and so providing these with a first *generic* insight, as well as yielding a *first* proof of the maximal paraconsistency of P^1 [22] (being conjunctive too), in its turn, subsuming its *axiomatic* maximal paraconsistency discovered in [22] and equally subsumed by either Proposition 5.2 or Corollary 6.19 below (in particular, Theorem 6.3 of [12]), consists in extending the latter beyond subclassical logics towards those with merely subclassical negation, in which case, contrary to the latter, the former is equally applicable to arbitrary three-valued expansions (cf. Corollary 4.6 in this connection) of logics under consideration, because expansions retain (weak) conjunction, subclassical negation and paraconsistency, but do not, generally speaking, inherit the property of being subclassical, and so the former, as opposed to the latter, covers *arbitrary three-valued expansions* of LP (including those of its three-valued expansion $\mathbb{S}_3$), HZ and P^1. After all, in view of Example 5.10 below, the stipulation of weak conjunctivity cannot be omitted in the formulation of Corollary 5.4. Likewise, the instance of $\mathbb{S}_5$ (cf. Paragraph [23]) shows that the reservation "three-valued" cannot be omitted in the formulations of Corollaries 4.6 and 5.4.

5.1. **Premaximal paraconsistency.** Let $C_{\frac{1}{2}}$ be the logic of $\mathcal{A}_{\frac{1}{2}}$.

Lemma 5.5. *Let $\mathcal{B} \in \mathrm{Mod}(C)$. Suppose C has a theorem and is a sublogic of $C_{\frac{1}{2}}$. Then, $\mathcal{B}$ is consistent iff it is $\sim$-paraconsistent. In particular, $\mathcal{A}_{\frac{1}{2}}$ (viz., $C_{\frac{1}{2}}$) is $\sim$-paraconsistent.*

Proof. The "if" part is immediate. Conversely, assume $\mathcal{B}$ is consistent. Then, by the structurality of C, applying the Σ-substitution extending $[x_i/x_0]_{i\in\omega}$ to any theorem of C, we conclude that there is some $\phi \in (\mathrm{Fm}^1_\Sigma \cap C(\varnothing))$, and so, as $\mathcal{A}_{\frac{1}{2}} \in \mathrm{Mod}(C)$, $\phi^{\mathfrak{A}}(a) = \frac{1}{2}$, for all $a \in A$. Take any $b \in (B \setminus D^{\mathcal{B}}) \neq \varnothing$, for $\mathcal{B}$ is consistent. Then, the submatrix $\mathcal{D}$ of $\mathcal{B}$ generated by $\{b\}$ is a finitely-generated consistent model of C. Hence, by Lemma 2.2, there are some set I and some submatrix $\mathcal{E} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{D}))$ of $\mathcal{A}^I$. Take any $e \in E \neq \varnothing$. Then, $\phi^{\mathfrak{E}}(e) = (I \times \{\frac{1}{2}\}) \in D^{\mathcal{E}}$, in which case $\sim^{\mathfrak{E}}\phi^{\mathfrak{E}}(e) \in D^{\mathcal{E}}$, and so $\mathcal{E}$, being consistent, for $\mathcal{D}$ is so, is $\sim$-paraconsistent. Thus, $\mathcal{B}$ is so, as required. □

Remark 5.6. Let $\mathcal{B}$ be a canonically $\sim$-[super-]classical Σ-matrix and C' the logic of $\mathcal{B}$. Suppose $\mathcal{B}$ (viz., C') is weakly $\veebar$-disjunctive. then, as it is false-singular, while $0 \notin D^{\mathcal{B}}$), whereas $\sim^{\mathfrak{B}}0 = 1 \in D^{\mathcal{B}}$, $x_0 \veebar \sim x_0$ is a theorem of C'. □

Theorem 5.7. *Suppose C has a proper $\sim$-paraconsistent extension. Then, the following hold:*

(i) *$C_{\frac{1}{2}}$ is the proper ($\sim$-para)consistent extension of C relatively axiomatized by* (4.2)*;*
(ii) *$C_{\frac{1}{2}}$ is maximally inferentially consistent (in particular, $\sim$-paraconsistent);*
(iii) *the following are equivalent:*
  a) *C has a theorem;*
  b) 2 *does not form a subalgebra of $\mathfrak{A}$;*
  c) *C is not $\sim$-subclassical;*
  d) *$C_{\frac{1}{2}}$ is the only proper ($\sim$-para)consistent extension of C;*
  e) *$C_{\frac{1}{2}}$ has no proper sublogic being a proper extension of C.*

In particular, C is premaximally $\sim$-paraconsistent iff it either is maximally $\sim$-paraconsistent or "is not $\sim$-subclassical"/"has a theorem (in particular, is weakly disjunctive [in particular, implicative])".

Proof. In that case, by Theorem 5.1(iii/iv/viii)⇒(i), $\mathcal{A}_{\frac{1}{2}} \in \mathrm{Mod}(C)$ is $\sim$-paraconsistent, while $\mathcal{A}$ has no ternary $\frac{1}{2}$-relative semi-conjunction, whereas $\{\frac{1}{2}\}|L_3$ forms a subalgebra of $\mathfrak{A}|\mathfrak{A}^2$, respectively (in particular, $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$).

(i) Then, (4.2), not being true in $\mathcal{A}$ under $[x_0/1]$, is true in $\mathcal{A}_{\frac{1}{2}}$. In this way, the logic of $\mathcal{A}_{\frac{1}{2}}$ is a proper ($\sim$-para)consistent extension of C satisfying (4.2). Conversely, consider any Σ-rule $\Gamma \vdash \phi$ not satisfied in the extension C' of C relatively axiomatized by (4.2), in which case, as $\sim[\Gamma] \subseteq C'(\Gamma)$, the Σ-rule $(\Gamma \cup \sim[\Gamma]) \vdash \phi$ is not satisfied in C', and so in its sublogic C. Then, there is some $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $h[\Gamma \cup \sim[\Gamma]] \subseteq D^{\mathcal{A}} = \{\frac{1}{2}, 1\} \not\ni h(\phi)$. In particular, $h(\phi) \neq \frac{1}{2}$. And what is more, for each $\psi \in \Gamma$, both $h(\psi) \in D^{\mathcal{A}}$ and $\sim^{\mathfrak{A}} h(\psi) = h(\sim\psi) \in D^{\mathcal{A}}$, in which case $h(\psi) = \frac{1}{2}$, for $\sim^{\mathfrak{A}} 1 = 0 \notin D^{\mathcal{A}}$, and so $h[\Gamma] \subseteq \{\frac{1}{2}\} = D^{\mathcal{A}_{\frac{1}{2}}} \not\ni h(\phi)$. Thus, $C' = C_{\frac{1}{2}}$.

(ii) Consider any inferentially consistent extension C'' of $C_{\frac{1}{2}}$, in which case $x_1 \notin T \triangleq C''(x_0) \ni x_0$. Then, by the structurality of C'', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C'' (in particular, of $C_{\frac{1}{2}}$), and so is its finitely-generated consistent truth-non-empty submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^2_{\Sigma}, T \cap \mathrm{Fm}^2_{\Sigma}\rangle$. Hence, by Lemma 2.2, there are some set I and some submatrix $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of $\mathcal{A}^I_{\frac{1}{2}}$, in which case, $\mathcal{D}$ is a consistent truth-non-empty model of C'', for $\mathcal{B}$ is so, and so $I \neq \varnothing$, while there are some $a \in D^{\mathcal{D}}$ and some $b \in (D \setminus D^{\mathcal{D}})$. Then, $D \ni a = (I \times \{\frac{1}{2}\}) \neq b$, in which case either $J \triangleq \{i \in I \mid \pi_i(b) = 1\}$ or $K \triangleq \{i \in I \mid \pi_i(b) = 0\}$ is non-empty. Given any $\bar{c} \in A^3$, set $(c_0 : c_1 : c_2) \triangleq ((J \times \{c_0\}) \cup (K \times \{c_1\}) \cup ((I \setminus (J \cup K)) \times \{c_2\})) \in A^I$. In this way, $D \ni a = (\frac{1}{2} : \frac{1}{2} : \frac{1}{2})$ and $D \ni b = (1 : 0 : \frac{1}{2})$, in which case $D \ni \sim^{\mathfrak{D}} b = (0 : 1 : \frac{1}{2})$. Consider the following complementary cases:

- $J \neq \varnothing \neq K$.
  Then, as $\{\frac{1}{2}\}|L_3$ forms a subalgebra of $\mathfrak{A}|\mathfrak{A}^2$, $\{\langle\langle x, y\rangle, (x : y : \frac{1}{2})\rangle \mid \langle x, y\rangle \in L_3\}$ is an embedding of $\mathcal{E} \triangleq (\mathcal{A}^2 \upharpoonright L_3)$ into $\mathcal{D}$, in which case, $\mathcal{E}$ is a model of C'', for $\mathcal{D}$ is so, and so is $\mathcal{A}_{\frac{1}{2}}$, for $(\pi_0 \upharpoonright L_3) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{A}_{\frac{1}{2}})$.
- $K = \varnothing$,
  in which case $J \neq \varnothing$, while $D \ni a = (\frac{1}{2} : \frac{1}{2} : \frac{1}{2})$, whereas $D \ni b = (0 : \frac{1}{2} : \frac{1}{2})$, and so $D \ni \sim^{\mathfrak{D}} b = (1 : \frac{1}{2} : \frac{1}{2})$. Then, as $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, $\{\langle x, (x : \frac{1}{2} : \frac{1}{2})\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}_{\frac{1}{2}}$ into $\mathcal{D}$, in which case $\mathcal{A}_{\frac{1}{2}}$ is a model of C'', for $\mathcal{D}$ is so.
- $J = \varnothing$,
  in which case $K \neq \varnothing$, while $D \ni a = (\frac{1}{2} : \frac{1}{2} : \frac{1}{2})$, whereas $D \ni b = (\frac{1}{2} : 0 : \frac{1}{2})$, and so $D \ni \sim^{\mathfrak{D}} b = (\frac{1}{2} : 1 : \frac{1}{2})$. Then, as $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, $\{\langle x, (\frac{1}{2} : x : \frac{1}{2})\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}_{\frac{1}{2}}$ into $\mathcal{D}$, in which case $\mathcal{A}_{\frac{1}{2}}$ is a model of C'', for $\mathcal{D}$ is so.

Thus, in any case, $\mathcal{A}_{\frac{1}{2}} \in \mathrm{Mod}(C'')$, and so $C'' = C_{\frac{1}{2}}$.

(iii) First, assume **a)** holds. Consider any consistent extension C''' of C, in which case $C'''(\varnothing) \supseteq C(\varnothing) \neq \varnothing$, and so, if C''' was inferentially inconsistent, then it, being structural, would be inconsistent, and the following complementary cases:

- (4.2) is satisfied in C''',
  in which case, by (i), C''' is an inferentially consistent extension of $C_{\frac{1}{2}}$, and so, by (ii), $C''' = C_{\frac{1}{2}}$.
- (4.2) is not satisfied in C''',
  in which case $\sim x_0 \notin T \triangleq C'''(x_0) \ni x_0$. Then, by the structurality of C''', $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C''' (in particular, of C), in which (4.2)

is not true under the diagonal Σ-substitution, in which case, by Lemma 5.5, $\mathcal{B}$, being consistent, is $\sim$-paraconsistent, and so, by Lemma 4.4, $\mathcal{A}$ is a model of C''', for $\mathcal{B}$ is so, in which case $C''' = C$.

Thus, by (i), **d)** holds.

Next, **d)**$\Rightarrow$**e)** is by the ($\sim$-para)consistency of $\mathcal{A}_{\frac{1}{2}}$, and so of any sublogic of $C_{\frac{1}{2}}$.

Now, let $\mathcal{B}$ be a $\sim$-classical model of C. Then, (5.1), being a logical consequence of $((2.9)[x_0/x_1, x_1/{\sim}x_0])/(4.2)$, is true in $\mathcal{B}/\mathcal{A}_{\frac{1}{2}}$, for (2.9)/(4.2) is so /"in view of (i)". However, it is not true in $\mathcal{A}$ under $[x_0/1, x_1/\frac{1}{2}]$. Moreover, by (2.11) with $n = 0$ and $m = 1$, (4.2) is not true in $\mathcal{B}$. In this way, by (i), the logic of $\{\mathcal{A}_{\frac{1}{2}}, \mathcal{B}\}$ is a proper extension/sublogic of $C_{/\frac{1}{2}}$. Thus, **e)**$\Rightarrow$**c)** holds.

Further, if 2 forms a subalgebra of $\mathfrak{A}$, then $\mathcal{A}{\restriction}2$ is a $\sim$-classical model of C. Therefore, **c)**$\Rightarrow$**b)** holds.

Finally, assume **b)** holds. Then, there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(1,0) = \frac{1}{2} = \varphi^{\mathfrak{A}}(\frac{1}{2},\frac{1}{2})$, for $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, in which case, if $\varphi^{\mathfrak{A}}(0,1)$ was equal to 0, then φ would be a ternary $\frac{1}{2}$-relative semi-conjunction for $\mathcal{A}$, and so $\varphi^{\mathfrak{A}}(0,1) \in D^{\mathcal{A}} \supseteq \{\varphi^{\mathfrak{A}}(1,0), \varphi^{\mathfrak{A}}(\frac{1}{2},\frac{1}{2})\}$. In this way, $\varphi[x_1/{\sim}x_0]$ is a theorem of C, and so **a)** holds.

After all, (ii), (iii)**c/a)**$\Rightarrow$**d)** /"as well as Lemma 3.3 and Remark 5.6" complete the argument. □

In this way, Theorem[s] 5.1(i)$\Leftrightarrow$(iv) [and 5.7(iii)**b)**$\Leftrightarrow$**d)**] provide an effective algebraic criterion the [pre]maximal $\sim$-paraconsistency of C. And what is more, by Theorem 5.7(iii)**a)**$\Rightarrow$**c)**, we have:

Corollary 5.8. *C is maximally $\sim$-paraconsistent, whenever it is $\sim$-subclassical and has a theorem.*

Then, combining Lemma 3.3, Theorem 4.1 [and the last assertion of Theorem 5.1] with Remark 5.6 and Corollary 5.8, we get the following "disjunctive" analogue of Corollary 5.4, being essentially beyond the scopes of the reference [Pyn 95b] of [13], and so becoming a one more substantial advance of the present study with regard to that one:

Corollary 5.9. *C is maximally $\sim$-paraconsistent, whenever it is $\sim$-subclassical and weakly disjunctive (in particular, implicative). In particular, any [three-valued expansion of any] weakly disjunctive (in particular, implicative) $\sim$-subclassical three-valued $\sim$-paraconsistent Σ-logic is maximally $\sim$-paraconsistent.*

This is *immediately* applicable to arbitrary (not necessarily $\sim$-subclassical) three-valued expansions of the implicative $\sim$-subclassical P^1 and HZ. On the other hand, as opposed to Corollary 5.4, the condition of being $\sim$-subclassical in the formulation of Corollary 5.9 (even without the reservation "weakly") is essential, as it follows from the optional version of:

Example 5.10. Let $\Sigma \triangleq (\Sigma_\sim[\cup\{\vee\}])$, $\sim^{\mathfrak{A}}\frac{1}{2} \triangleq \frac{1}{2}$ [and:

$$(a \vee^{\mathfrak{A}} b) \triangleq \begin{cases} a & \text{if } a = b, \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

for all $a, b \in A$] in which case [(2.2) with $i = 0$, (2.3) (for $\veebar$ is a symmetric formula for $\mathcal{A}$) and (2.4) (for $\veebar$ is an idempotent formula for $\mathcal{A}$) are true in $\mathcal{A}$, and so, by Lemma 3.2, C is $\vee$-disjunctive, while] 2 does [not] form a subalgebra of $\mathfrak{A}$ [for $(0 \vee^{\mathfrak{A}} 1) = \frac{1}{2} \notin 2$]. Then, L_3 forms a subalgebra of $\mathfrak{A}^2$, in which case, by

Theorem 5.1(ii/iii)⇒(iv), $C/\mathcal{A}$ has "a proper non-$\sim$-subclassical $\sim$-paraconsistent (in particular, {inferentially-}consistent) extension"/"no binary semi-conjunction", and so C is [not] $\sim$-subclassical, in view of Theorem 5.7(iii)**b)**⇔**c)**. □

6. Classical extensions

A *quasi-negation for/of* $\mathcal{A}$ is any $\kappa \in \mathrm{Fm}^1_\Sigma$ such that $\kappa^{\mathfrak{A}}[\{\frac{1}{2},1\}] \subseteq \{0,\frac{1}{2}\}$. (Clearly, $\sim$ {resp., $[\sim]c$, where $c \in \Sigma$ is nullary} is a quasi-negation for $\mathcal{A}$, whenever $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$ {resp., $c^{\mathfrak{A}} \in ([A\setminus]\{0,\frac{1}{2}\})$}.) Likewise, a *ternary equalizer for/of* $\mathcal{A}$ is any $\varepsilon \in \mathrm{Fm}^3_\Sigma$ such that $\varepsilon^{\mathfrak{A}}(0,1,1) = \varepsilon^{\mathfrak{A}}(1,0,\frac{1}{2})$. (Clearly, any binary semi-conjunction for $\mathcal{A}$ as well as any nullary connective of Σ is a ternary equalizer for $\mathcal{A}$.) Note that $\mathcal{A}$ has a quasi-negation [resp., a ternary equalizer] iff the carrier of the subalgebra of $\mathfrak{A}^2$ generated by $\{\langle 1,\frac{1}{2}\rangle\}[\cup L_2]$ is not disjoint with $\{0,\frac{1}{2}\}^2$ [resp., Δ_A]) that yields an effective procedure of verifying it.

Lemma 6.1 (2nd Key Lemma)**.** *Let I be a finite set, $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and $\mathcal{D}$ a consistent non-$\sim$-paraconsistent subdirect product of it. Suppose either $\mathcal{D}$ is $\sim$-negative or $\mathcal{A}$ is either weakly conjunctive or both weakly disjunctive and truth-symmetric, or both either 2 forms a subalgebra of $\mathfrak{A}$ or L_4 forms a subalgebra of $\mathfrak{A}^2$ or $\mathcal{A}$ has a ternary equalizer, and either $\mathcal{A}$ has a binary semi-conjunction or both $\mathcal{D}$ is truth-non-empty and $\mathcal{A}$ has a quasi-negation. Then, the following hold:*

(i) *if 2 forms a subalgebra of $\mathfrak{A}$, then $\mathcal{A}{\restriction}2$ is embeddable into $\mathcal{D}$;*
(ii) *if 2 does not form a subalgebra of $\mathfrak{A}$, then neither $\mathcal{A}$ is weakly conjunctive nor $\mathcal{A}$ is truth-symmetric no $\mathcal{D}$ is disjunctive, while L_4 forms a subalgebra of $\mathfrak{A}^2$, whereas $\mathcal{A}^2{\restriction}L_4$ is embeddable into $\mathcal{D}$.*

Proof. In that case, we first have:

$$(I \times \{\tfrac{1}{2}\}) \notin D, \tag{6.1}$$

for, otherwise, we would get $\{I \times \{\frac{1}{2}\}, \sim^{\mathfrak{D}}(I \times \{\frac{1}{2}\})\} \subseteq D^{\mathcal{D}}$, and so $\mathcal{D}$, being consistent, would be $\sim$-paraconsistent. And what is more, $\mathcal{D} \in \mathrm{Mod}(C)$ is then truth-non-empty, in view of Remarks 5.3, 5.6 and:

Claim 6.2. *Let φ be a binary semi-conjunction for $\mathcal{A}$. Then, C has a theorem.*

Proof. Let $\mathcal{D}$ be the submatrix of $\mathcal{A}^3$ generated by $a \triangleq (10\frac{1}{2})$. Consider the following exhaustive cases:

- $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$.
  Then, $D \ni b \triangleq \sim^{\mathfrak{D}}a = (01\frac{1}{2})$. Let $x \triangleq \varphi^{\mathfrak{A}}(\frac{1}{2},\frac{1}{2}) \in A$. Consider the following exhaustive subcases:
  - $x = \frac{1}{2}$.
    Then, $D \ni c \triangleq \varphi^{\mathfrak{D}}(a,b) = (00\frac{1}{2})$. In this way, $D \ni d \triangleq \sim^{\mathfrak{D}}c = (11\frac{1}{2}) \in (D^{\mathcal{A}})^3$.
  - $x = 0$.
    Then, $D \ni c \triangleq \varphi^{\mathfrak{D}}(a,b) = (000)$. In this way, $D \ni d \triangleq \sim^{\mathfrak{D}}c = (111) \in (D^{\mathcal{A}})^3$.
  - $x = 1$.
    Then, $D \ni c \triangleq \varphi^{\mathfrak{D}}(a,b) = (001)$, in which case $D \ni \sim^{\mathfrak{D}}c = (110)$, and so $D \ni d \triangleq \sim^{\mathfrak{D}}\varphi^{\mathfrak{D}}(c,\sim^{\mathfrak{D}}c) = (111) \in (D^{\mathcal{A}})^3$.
- $\sim^{\mathfrak{A}}\frac{1}{2} = 1$.
  Then, $D \ni b \triangleq \sim^{\mathfrak{D}}a = (011)$, in which case $D \ni \sim^{\mathfrak{D}}b = (100)$, and so $D \ni d \triangleq \sim^{\mathfrak{D}}\varphi^{\mathfrak{D}}(b,\sim^{\mathfrak{D}}b) = (111) \in (D^{\mathcal{A}})^3$.

Thus, anyway, $d \in ((D^{\mathcal{A}})^3 \cap D)$, in which case there is some $\phi \in \mathrm{Fm}^1_\Sigma$ such that $d = \phi^{\mathfrak{A}^3}(10\frac{1}{2})$, and so, since $(\mathrm{img}\, a) = A$, $\phi \in C(\varnothing)$, as required. □

Take any $a \in D^{\mathcal{D}} \subseteq \{\frac{1}{2}, 1\}^I$, in which case, by (6.1), $J \triangleq \{i \in I \mid \pi_i(a) = 1\} \neq \varnothing$, and so $D \ni {\sim}^{\mathfrak{D}} a \notin D^{\mathcal{D}}$. Given any $\bar{e} \in A^2$, set $(e_0 : e_1) \triangleq ((J \times \{e_0\}) \cup ((I \setminus J) \times \{e_1\})) \in A^I$, in which case $a = (1 : \frac{1}{2})$. Consider the following complementary cases:

(i) 2 forms a subalgebra of $\mathfrak{A}$.
Consider the following complementary subcases:
- $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$,
in which case ${\sim}^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$, and so $(1 : \frac{1}{2}) = a \in D \ni {\sim}^{\mathfrak{D}} a = (0 : \frac{1}{2})$. Then, as $J \neq \varnothing$, $\{\langle k, (k : \frac{1}{2})\rangle \mid k \in 2\}$ is an embedding of $\mathcal{A}{\restriction}2$ into $\mathcal{D}$.
- $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$,
in which case there is some $\psi \in \mathrm{Fm}^1_\Sigma$ such that $\psi^{\mathfrak{A}}(\frac{1}{2}) \in 2$, and so, as 2 forms a subalgebra of $\mathfrak{A}$, $\psi^{\mathfrak{A}} : A \to 2$ is not injective, for $|A| = 3 \nleq 2 = |2|$. Consider the following exhaustive subsubcases:
  - ${\sim}^{\mathfrak{A}} \frac{1}{2} = 1$,
  in which case ${\sim}^{\mathfrak{D}} a = (0 : 1) \notin D^{\mathcal{D}}$, and so ${\sim}^{\mathfrak{D}}{\sim}^{\mathfrak{D}} a = (1 : 0)$. Consider the following exhaustive subsubsubcases:
    * $\mathcal{D}$ is ${\sim}$-negative,
    in which case ${\sim}^{\mathfrak{D}}{\sim}^{\mathfrak{D}} a \in D^{\mathcal{D}}$, and so $J = I$. Then, $D \ni a = (I \times \{1\})$.
    * $\mathcal{A}$ is weakly conjunctive,
    in which case, by Lemma 3.1, $(I \times \{0\}) \in D$, and so $D \ni {\sim}^{\mathfrak{D}}(I \times \{0\}) = (I \times \{1\})$.
    * $\mathcal{A}$ is weakly $\diamond$-disjunctive,
    in which case $(0 \diamond^{\mathfrak{A}} 1) = 1 = (1 \diamond^{\mathfrak{A}} 0)$, for $D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$, while 2 forms a subalgebra of $\mathfrak{A}$, and so $D \ni ({\sim}^{\mathfrak{D}} a \diamond^{\mathfrak{D}} {\sim}^{\mathfrak{D}}{\sim}^{\mathfrak{D}} a) = (1 : 1) = (I \times \{1\})$.
    * $\mathcal{A}$ has a binary semi-conjunction φ,
    in which case $D \ni {\sim}^{\mathfrak{D}}\varphi^{\mathfrak{D}}({\sim}^{\mathfrak{D}} a, {\sim}^{\mathfrak{D}}{\sim}^{\mathfrak{D}} a) = (1 : 1) = (I \times \{1\})$.
    * $\mathcal{A}$ has a quasi-negation κ,
    in which case $D \ni \kappa^{\mathfrak{D}}(a) \in \{0, \frac{1}{2}\}^I$, and so $D \ni {\sim}^{\mathfrak{D}}\kappa^{\mathfrak{D}}(a) = (I \times \{1\})$.
  - ${\sim}^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$,
  in which case ${\sim}^{\mathfrak{D}} a = (0 : \frac{1}{2})$. Then, there are some distinct $\imath, \jmath \in A$ such that $\psi^{\mathfrak{A}}(\imath) = \psi^{\mathfrak{A}}(\jmath)$, in which case $\{\imath, \jmath\} = (A \setminus \{\ell\})$, for some $\ell \in A$, and so we have the following three exhaustive subsubsubcases:
    * $\ell = 1$,
    in which case $\{\imath, \jmath\} = \{\frac{1}{2}, 0\}$, and so $\psi^{\mathfrak{A}}(\frac{1}{2}) = \psi^{\mathfrak{A}}(0)$. Then, $(I \times \{1\}) \in \{\psi^{\mathfrak{D}}({\sim}^{\mathfrak{D}} a), {\sim}^{\mathfrak{D}}\psi^{\mathfrak{D}}({\sim}^{\mathfrak{D}} a)\} \subseteq D$.
    * $\ell = 0$,
    in which case $\{\imath, \jmath\} = \{\frac{1}{2}, 1\}$, and so $\psi^{\mathfrak{A}}(\frac{1}{2}) = \psi^{\mathfrak{A}}(1)$. Then, $(I \times \{1\}) \in \{\psi^{\mathfrak{D}}(a), {\sim}^{\mathfrak{D}}\psi^{\mathfrak{D}}(a)\} \subseteq D$.
    * $\ell = \frac{1}{2}$,
    in which case $\{\imath, \jmath\} = 2$, and so $\psi^{\mathfrak{A}}(1) = \psi^{\mathfrak{A}}(0)$. Then, $(I \times \{1\}) \in \{\psi^{\mathfrak{D}}(\psi^{\mathfrak{D}}(a)), {\sim}^{\mathfrak{D}}\psi^{\mathfrak{D}}(\psi^{\mathfrak{D}}(a))\} \subseteq D$.

  Thus, anyway, $(I \times \{1\}) \in D$, in which case $D \ni {\sim}^{\mathfrak{D}}(I \times \{1\}) = (I \times \{0\})$, and so, since $I \supseteq J \neq \varnothing$, $\{\langle k, I \times \{k\}\rangle \mid k \in 2\}$ is an embedding of $\mathcal{A}{\restriction}2$ into $\mathcal{D}$.

(ii) 2 does not form a subalgebra of $\mathfrak{A}$,
in which case there is some $\phi \in \mathrm{Fm}^2_\Sigma$ such that $\phi^{\mathfrak{A}}(1,0) = \frac{1}{2}$, and so

$$(I \times \{k\}) \notin D, \tag{6.2}$$

for any $k \in 2$, because, otherwise, D would contain $\sim^{\mathfrak{D}}(I \times \{k\}) = (I \times \{1-k\})$, and so, since $2 = \{k, 1-k\}$, D would contain $\phi^{\mathfrak{D}}(I \times \{1\}, I \times \{0\}) = (I \times \{\frac{1}{2}\})$, contrary to (6.1). Then, by (6.2) with $k=1$, $I \neq J$. Moreover, by Lemma 3.1 and (6.2) with $k=0$, $\mathcal{A}$, being false-singular, is not weakly conjunctive. Let $\eta \triangleq (\phi[x_1/{\sim}x_0]) \in \mathrm{Fm}^1_\Sigma$, in which case $\eta^{\mathfrak{A}}(1) = \phi^{\mathfrak{A}}(1,0) = \frac{1}{2}$, and so, by (6.1), $\eta^{\mathfrak{A}}(\frac{1}{2}) \in 2$, for $\eta^{\mathfrak{D}}(a) \in D$. And what is more, if it did hold that both $\sim^{\mathfrak{A}}\frac{1}{2} = 1$ and $\eta^{\mathfrak{A}}(\frac{1}{2}) = 0$, then D would contain $\sim^{\mathfrak{D}}\eta^{\mathfrak{D}}(a) = (1:1) = (I \times \{1\})$, contrary to (6.2) with $k=1$. Hence, $b \triangleq (\frac{1}{2}:1) \in \{\eta^{\mathfrak{D}}(a), \sim^{\mathfrak{D}}\eta^{\mathfrak{D}}(a)\} \subseteq D$. Therefore, in particular, if $\mathcal{A}$ had a truth-symmetric formula τ, then we would have $\tau^{\mathfrak{A}}(\frac{1}{2},1) = \tau^{\mathfrak{A}}(1,\frac{1}{2})$, in which case we would get $D \ni \tau^{\mathfrak{D}}(a,b) = (\tau^{\mathfrak{A}}(1,\frac{1}{2}) : \tau^{\mathfrak{A}}(\frac{1}{2},1)) = (\tau^{\mathfrak{A}}(\frac{1}{2},1) : \tau^{\mathfrak{A}}(\frac{1}{2},1))$, contrary to (6.1) and (6.2), and so $\mathcal{A}$ is not truth-symmetric. First, let us prove that $c \triangleq (0:\frac{1}{2}) \in D \ni d \triangleq (\frac{1}{2}:0)$. For consider the following exhaustive subcases:

- $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$,
  in which case $c = \sim^{\mathfrak{D}}a \in D \ni \sim^{\mathfrak{D}}b = d$.
- $\sim^{\mathfrak{A}}\frac{1}{2} = 1$,
  in which case $D^{\mathcal{B}} \not\ni \sim^{\mathfrak{D}}a = (0:1) \in D \ni \sim^{\mathfrak{D}}\sim^{\mathfrak{D}}a = (1:0) \notin D^{\mathcal{D}}$, for $I \neq J$, and so $\mathcal{D}$ is not $\sim$-negative, while $\mathcal{A}$ has no binary semi-conjunction, because, otherwise, we would have $D \ni \xi^{\mathfrak{D}}(\sim^{\mathfrak{D}}a, \sim^{\mathfrak{D}}\sim^{\mathfrak{D}}a)$ $= (0:0) = (I \times \{0\})$, for any binary semi-conjunction ξ for $\mathcal{A}$, contrary to (6.2) with $k=0$. Hence, $\mathcal{A}$, being neither weakly conjunctive nor truth-symmetric, has a quasi-negation κ, and so, since $\kappa^{\mathfrak{D}}(a) \in D$, by (6.1) and (6.2) with $k=0$, we have $\kappa^{\mathfrak{A}}(1) \neq \kappa^{\mathfrak{A}}(\frac{1}{2})$. Consider the following exhaustive subsubcases:
  - $\kappa^{\mathfrak{A}}(1) = 0$,
    in which case $\kappa^{\mathfrak{A}}(\frac{1}{2}) = \frac{1}{2}$, and so $c = \kappa^{\mathfrak{A}}(a) \in D \ni \kappa^{\mathfrak{A}}(b) = d$.
  - $\kappa^{\mathfrak{A}}(1) = \frac{1}{2}$,
    in which case $\kappa^{\mathfrak{A}}(\frac{1}{2}) = 0$, and so $d = \kappa^{\mathfrak{A}}(a) \in D \ni \kappa^{\mathfrak{A}}(b) = c$.

Thus, anyway, $D^{\mathcal{D}}$ is disjoint with $\{c,d\} \subseteq D$, for $I \neq J \neq \varnothing$. Next, we prove, by contradiction, that $\mathcal{D}$ is not $\veebar$-disjunctive. For suppose $\mathcal{D}$ is $\veebar$-disjunctive, in which case, as $a \in D^{\mathcal{D}}$, we have $\{a \veebar^{\mathfrak{D}} d, d \veebar^{\mathfrak{D}} a\} \subseteq D^{\mathcal{D}}$, and so, since $I \neq J$, we get $\{\frac{1}{2} \veebar^{\mathfrak{A}} 0, 0 \veebar^{\mathfrak{A}} \frac{1}{2}\} \subseteq D^{\mathcal{A}}$. Then, $(c \veebar^{\mathfrak{D}} d) \in D^{\mathcal{D}}$, in which case, by the $\veebar$-disjunctivity of $\mathcal{D}$, $(\{c,d\} \cap D^{\mathcal{D}}) = \varnothing$, and so this contradiction shows that $\mathcal{D}$ is not $\veebar$-disjunctive. Further, we prove, by contradiction, that L_4 forms a subalgebra of $\mathfrak{A}^2$. For suppose L_4 does not form a subalgebra of $\mathfrak{A}^2$, in which case there is some $\zeta \in \mathrm{Fm}^4_\Sigma$ such that $\zeta^{\mathfrak{A}^2}(\langle 1,\frac{1}{2}\rangle, \langle \frac{1}{2},1\rangle, \langle 0,\frac{1}{2}\rangle, \langle \frac{1}{2},0\rangle) \in (A^2 \setminus L_4) = (\Delta_2 \cup L_3)$, and so $D \ni e \triangleq \zeta^{\mathfrak{D}}(a,b,c,d) = (x:y)$, where $\langle x,y\rangle \in (\Delta_2 \cup L_3)$. Then, by (6.1) and (6.2), $\langle x,y\rangle \in L_2$, in which case $0 \in \{x,y\}$, and so $e \in (D \setminus D^{\mathcal{D}}) \ni (y:x) = \sim^{\mathfrak{D}}e$, for $I \neq J \neq \varnothing$. Hence, $\mathcal{D}$ is not $\sim$-negative, in which case $\mathcal{A}$, being neither weakly conjunctive nor truth-symmetric, has a ternary equalizer ε, and so $D \ni \varepsilon^{\mathfrak{D}}((0:1),(1:0),a) = (z:z)$, where $z \triangleq \varepsilon^{\mathfrak{A}}(0,1,1) \in A$, for $\{(0:1),(1:0)\} = \{e, \sim^{\mathfrak{D}}e\} \subseteq D \ni a$. This contradicts to (6.1) and (6.2). Thus, L_4 forms a subalgebra of $\mathfrak{A}^2$, while $\{(u:v) \mid \langle u,v\rangle \in L_4\} = \{a,b,c,d\} \subseteq D$, whereas $I \neq J \neq \varnothing$. Therefore, $\{\langle\langle u,v\rangle,(u:v)\rangle \mid \langle u,v\rangle \in L_4\}$ is an embedding of $\mathcal{A}^2{\restriction}L_4$ into $\mathcal{D}$, as required. $\square$

Corollary 6.3. *Let $\mathcal{B}$ be a $\sim$-classical model of C. Then, the following hold:*
(i) *if* 2 *forms a subalgebra of* $\mathfrak{A}$*, then* $\mathcal{A}\restriction 2$ *is isomorphic to* $\mathcal{B}$*;*
(ii) *if* 2 *does not form a subalgebra of* $\mathfrak{A}$*, then both* $\mathcal{B}$ *is not disjunctive,* $\mathcal{A}$ *is neither weakly conjunctive nor truth-symmetric, and* C *is maximally* $\sim$*-paraconsistent, while* L_4 *forms a subalgebra of* $\mathfrak{A}^2$*, whereas* $\theta^{\mathcal{A}^2\restriction L_4} \in \mathrm{Con}(\mathfrak{A}^2\restriction L_4)$*,* $\langle \chi^{\mathcal{A}^2\restriction L_4}[\mathfrak{A}^2\restriction L_4], \{1\}\rangle$ *being isomorphic to* $\mathcal{B}$*.*

Proof. Then, $\mathcal{B}$ is finite and simple. Therefore, by Lemma 2.2, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it and some $g \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D}, \mathcal{B})$, in which case $\mathcal{D}$ is $\sim$-negative, for $\mathcal{B}$ is so, and so both consistent and not $\sim$-paraconsistent. Consider the following complementary cases:
(i) 2 forms a subalgebra of $\mathfrak{A}$.
Then, by Lemma 6.1(i), there is some embedding e of $\mathcal{A}\restriction 2$ into $\mathcal{D}$, in which case $e \circ g$ is that into $\mathcal{B}$, for $\mathcal{A}\restriction 2$, being $\sim$-classical, is simple, and so is an isomorphism from $\mathcal{A}\restriction 2$ onto $\mathcal{B}$, for this, being $\sim$-classical, has no proper submatrix.
(ii) 2 does not form a subalgebra of $\mathfrak{A}$.
Then, by Theorem 5.7(iii)**b)**$\Rightarrow$**c)** and Lemma 6.1(ii), both $\mathcal{D}$ is non-disjunctive (and so is $\mathcal{B}$), $\mathcal{A}$ is neither weakly conjunctive nor truth-symmetric, and C is maximally $\sim$-paraconsistent, while L_4 forms a subalgebra of $\mathfrak{A}^2$, whereas there is some embedding e of of $\mathcal{E} \triangleq (\mathcal{A}^2\restriction L_4)$ into $\mathcal{D}$. Let $\mathcal{F}$ be the canonically $\sim$-classical Σ-matrix isomorphic to $\mathcal{B}$. Take any isomorphism f from $\mathcal{B}$ onto $\mathcal{F}$. Then, $h \triangleq ((e \circ g) \circ f) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{E}, \mathcal{F})$, for $\mathcal{F}$, being $\sim$-classical, has no proper submatrix, in which case $h = \chi^{\mathcal{E}}$, and so $\theta^{\mathcal{E}} = (\ker h) \in \mathrm{Con}(\mathfrak{E})$, while $\mathfrak{F} = h[\mathfrak{E}]$. □

By [Lemma 3.2 and] Corollary 6.3, we immediately get:

Theorem 6.4. *C has a [$\veebar$-disjunctive] $\sim$-classical extension iff either of the following [but (ii)] holds:*
(i) 2 *forms a subalgebra of* $\mathfrak{A}$ *[with* $\veebar$*-disjunctive* $\mathcal{A}\restriction 2$*], in which case* $\mathcal{A}\restriction 2$ *is a canonically* $\sim$*-classical model of* C *isomorphic (and so equal) to any (canonically)* $\sim$*-classical model of* C*, and so defines a unique* $\sim$*-classical extension of* C*;*
(ii) 2 *does not form a subalgebra of* $\mathfrak{A}$*, while* L_4 *forms a subalgebra of* $\mathfrak{A}^2$*, whereas* $\theta^{\mathcal{A}^2\restriction L_4} \in \mathrm{Con}(\mathfrak{A}^2\restriction L_4)$*, in which case* $\langle \chi^{\mathfrak{A}^2\restriction L_4}[\mathfrak{A}^2\restriction L_4], \{1\}\rangle$ *is a canonicaly* $\sim$*-classical model of* C *isomorphic (and so equal) to any (canonically)* $\sim$*-classical model of* C*, and so defines a unique* $\sim$*-classical extension of* C*, while* $\mathcal{A}$ *is neither weakly conjunctive nor truth-symmetric, whereas* C *is maximally* $\sim$*-paraconsistent.*

According to Theorem 6.4, providing C is $\sim$-subclassical, there is a unique (canonically) $\sim$-classical extension (resp., model) of C to be denoted by C^{PC} (resp., $\mathcal{A}_{\mathrm{PC}}$ that defines C^{PC}) and referred to as *characteristic of|for C*.

It is remarkable that the $\veebar$-disjunctivity of C is not required in the []-optional version of Theorem 6.4, making this the right characterization of C's being *genuinely* $\sim$-subclassical in the sense of having a functionally complete $\sim$-classical extension. And what is more, by Lemmas 3.2, 3.3 and Theorem 6.4, we have:

Corollary 6.5. *[Suppose $\mathcal{A}$ is either weakly conjunctive or strongly disjunctive (in particular, implicative).] Then, C is $\sim$-subclassical if[f] 2 forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}\restriction 2$ is a canonically $\sim$-classical model of C isomorphic (and so equal) to any (canonically) $\sim$-classical model of C, and so defines a unique $\sim$-classical extension of C.*

The []-optional stipulation(s) in the formulation of Corollary 6.5 (resp., Theorem 6.4) cannot be omitted {or, even, “weakened”}, because of existence of three-valued {even, weakly disjunctive} $\sim$-paraconsistent $\sim$-subclassical Σ-logics with subclassical negation $\sim$ with non-classicaly-hereditary characteristic matrices, as it ensues from:

Example 6.6. Let $\Sigma \triangleq \{\amalg, \sim\}$ with binary $\amalg$, $\mathcal{B}$ the canonically $\sim$-classical Σ-matrix with $(j \amalg^{\mathfrak{B}} k) \triangleq 1$, for all $j, k \in 2$, $\sim^{\mathfrak{A}} \frac{1}{2} \triangleq \frac{1}{2}$ and

$$(a \amalg^{\mathfrak{A}} b) \triangleq \begin{cases} 1 & \text{if } a = \frac{1}{2}, \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

for all $a, b \in A$, in which case $\mathcal{A}$ is weakly $\amalg$-disjunctive, for $(\mathrm{img}\, \amalg^{\mathfrak{A}}) \subseteq D^{\mathcal{A}}$, and so is C. Then, we have:

$$\begin{array}{rcl} (\langle \frac{1}{2}, a \rangle \amalg^{\mathfrak{A}^2} \langle b, \frac{1}{2} \rangle) & = & \langle 1, \frac{1}{2} \rangle \in L_4, \\ (\langle b, \frac{1}{2} \rangle \amalg^{\mathfrak{A}^2} \langle \frac{1}{2}, a \rangle) & = & \langle \frac{1}{2}, 1 \rangle \in L_4, \\ (\langle \frac{1}{2}, a \rangle \amalg^{\mathfrak{A}^2} \langle \frac{1}{2}, b \rangle) & = & \langle 1, \frac{1}{2} \rangle \in L_4, \\ (\langle a, \frac{1}{2} \rangle \amalg^{\mathfrak{A}^2} \langle b, \frac{1}{2} \rangle) & = & \langle \frac{1}{2}, 1 \rangle \in L_4, \end{array}$$

for all $a, b \in 2$. Hence, L_4 forms a subalgebra of $\mathfrak{A}^2$, while $\chi^{\mathcal{A}^2 \restriction L_4} \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}^2 \restriction L_4, \mathcal{B})$, in which case $\mathcal{B} \in \mathrm{Mod}(C)$, and so C is $\sim$-subclassical. However, $(0 \amalg^{\mathfrak{A}} 1) = \frac{1}{2}$, in which case 2 does not form a subalgebra of $\mathfrak{A}$, and so, by Corollary 6.5, C is neither disjunctive nor weakly conjunctive. □

Corollary 6.7. *Suppose C is $\veebar$-disjunctive. Then, $C^{\mathrm{R}} = C^{\mathrm{PC}}$, if C is $\sim$-subclassical, and C^{R} is inconsistent, otherwise. In particular, C^{R} is consistent iff C is $\sim$-subclassical.*

Proof. In that case, by Corollary 6.5, $\mathbf{S}^{\mathrm{NP}}_{*}(\mathcal{A}) = (\{\mathcal{A} \restriction 2\}[\cap \varnothing])$, whenever C is [not] $\sim$-subclassical, for $\mathcal{A}$ is $\sim$-paraconsistent, while $\frac{1}{2} \in D^{\mathcal{A}}$. In this way, Theorem 3.5 and Corollary 6.5 complete the argument. □

Theorem 6.8. *[Providing $\mathcal{A}$ is either weakly (and strongly) $\diamond$-conjunctive or both truth-symmetric and weakly (as well as strongly) $\diamond$-disjunctive] (iii)⇔(ii)⇒(iv)⇒(i) [⇒(ii)], where:*

(i) *C^{NP} is consistent;*
(ii) *C is $\sim$-subclassical;*
(iii) *C^{NP} is [(properly)] $\sim$-subclassical;*
(iv) *C^{NP} is axiomatically-equivalent to C.*

Proof. First, (iv)⇒(i) is by the consistency of C (viz., $\mathcal{A}$). Next, (iii)⇒(ii) is by the fact that $C \subseteq C^{\mathrm{NP}}$.

Further, assume (ii) holds. Let $\mathcal{B}$ be a $\sim$-classical model of C, in which case $\{\mathcal{A}\times\}\mathcal{B}$ is a non-$\sim$-paraconsistent one, and so is a model of C^{NP}. In particular, C^{NP} is a sublogic of the logic of $\mathcal{B}$. And what is more, $(\pi_0 \restriction (A \times B)) \in \hom^{\mathrm{S}}(\mathcal{A} \times \mathcal{B}, \mathcal{A})$. Hence, (iv) holds. [(Now, consider the following complementary cases:

- $\sim^{\mathfrak{A}} \frac{1}{2} = 1$,
  in which case the Σ-rule $x_0 \vdash \sim\sim x_0$, being true in $\mathcal{B}$, for this is $\sim$-negative, is not true in $(\mathcal{A} \times \mathcal{B}) \in \mathrm{Mod}(C^{\mathrm{NP}})$ under $[x_0/\langle \frac{1}{2}, 1 \rangle]$, and so is not satisfied in C^{NP}.
- $\sim^{\mathfrak{A}} \frac{1}{2} \neq 1$,
  in which case $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$. Then, $\mathcal{B} \in \mathrm{Mod}(C)$ is $\overline{\wedge}$-conjunctive/-disjunctive,

for C is so /(cf. Lemma 3.2), in which case it, being $\sim$-negative, is $\veebar$-disjunctive, where $\veebar \triangleq (\diamond^{\sim}/\diamond)$, and so $\sqsupset$-implicative, where $\sqsupset = \sqsupset_{\veebar}^{\sim}$. Moreover, by Corollary 6.5, 2 forms a subalgebra of $\mathfrak{A}$. And what is more, by the $\diamond$-conjunctivity/-disjunctivity of $\mathcal{A}$, we have $((\frac{1}{2}/(\frac{1}{2}|0) \diamond^{\mathfrak{A}} (1/(0|1))) \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$. Consider the following complementary subcases:

- either $\mathcal{A}$ is not $\diamond$-conjunctive (in which case it is $\diamond$-disjunctive) or, otherwise, $(\frac{1}{2} \diamond^{\mathfrak{A}} 1) = \frac{1}{2}$,
  in which case the Σ-rule (2.5), being true in $\mathcal{B}$, for this is $\sqsupset$-implicative, is not true in $(\mathcal{A} \times \mathcal{B}) \in \mathrm{Mod}(C^{\mathrm{NP}})$ under $[x_0/\langle\frac{1}{2}, 1\rangle, x_1/\langle 0, 1\rangle]$, and so is not satisfied in C^{NP}.
- both $\mathcal{A}$ is $\diamond$-conjunctive and $(\frac{1}{2} \diamond^{\mathfrak{A}} 1) = 1$,
  in which case the Σ-rule (2.2), being true in $\mathcal{B}$, for this is $\veebar$-disjunctive, is not true in $(\mathcal{A} \times \mathcal{B}) \in \mathrm{Mod}(C^{\mathrm{NP}})$ under $[x_0/\langle\frac{1}{2}, 1\rangle, x_1/\langle 0, 1\rangle]$, and so is not satisfied in C^{NP}.

Thus, anyway, there is a Σ-rule, which is true in $\mathcal{B}$ but is not satisfied in C^{NP}. Therefore, C^{NP} is a proper sublogic of the logic of $\mathcal{B}$.)] Hence, (iii) holds.

[Finally, assume (i) holds, in which case, by the structurality of C^{NP}, $x_0 \notin T \triangleq C^{\mathrm{NP}}(\varnothing)$, whereas $\langle \mathfrak{Fm}_{\Sigma}^{\omega}, T\rangle$ is a model of C^{NP} (in particular, of C), and so is its consistent finitely-generated submatrix $\mathcal{B}' \triangleq \langle \mathfrak{Fm}_{\Sigma}^{1}, T \cap \mathrm{Fm}_{\Sigma}^{1}\rangle$. Hence, by Lemma 2.2, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}'))$ of it, in which case $\mathcal{D}$ is a consistent model of C^{NP}, for $\mathcal{B}'$ is so, and so is non-$\sim$-paraconsistent, for C^{NP} is so. In this way, (ii) is by Lemma 6.1(ii) and Theorem 6.4, as required.] □

Corollary 6.9. *[Providing $\mathcal{A}$ (viz., C) is weakly conjunctive] C has a proper inferentially-consistent [resp., consistent] extension if[f] it is $\sim$-subclassical.*

Proof. The "if" part is by the inferential consistency of $\sim$-classical Σ-logics. [Conversely, consider any proper consistent extension C' of C, in which case, by Corollary 5.4, C' is an extension of C^{NP}, and so this is consistent. In this way, Theorem 6.8(i)⇒(ii) completes the argument.] □

The optional stipilation of weak conjunctivity can be neither omitted nor even replaced by those of disjunctivity and symmetry in the optional version of Corollary 6.9, in view of the optional version of Example 5.10.

In Subsection 6.2, we obtain some more similar characterizations of C's being $\sim$-subclassical. Before (in the next subsection), we study a closely related issue.

6.1. **Non-subclassical extensions.** If C is not $\sim$-subclassical, then it, being (inferentially-)consistent, for $\mathcal{A}$ is so, is its own (inferentially-)consistent non-$\sim$-subclassical extension. Here, we explore the opposite case.

6.1.1. *Theorems versus binary semi-conjunctions and consistent non-subclassical extensions of subclassical 3VPLSN.*

Lemma 6.10. *Suppose C is $\sim$-subclassical. Then, the following are equivalent:*

(i) *C^{PC} has a theorem;*
(ii) *$\mathcal{A}$ has a binary semi-conjunction;*
(iii) *$L_{2[+6]}$ does not form a subalgebra of $(\mathfrak{A}^{[2]})^2$, whenever 2 does [not] form a subalgebra of $\mathfrak{A}$, where $L_8 \triangleq (L_4^2 \setminus (\bigcup_{i\in 2}\{\langle\frac{1}{2}, i\rangle, \langle i, \frac{1}{2}\rangle\}^2))$.*

Proof. We start from proving:

Claim 6.11. *Let $\mathcal{B}$ be a canonically $\sim$-classical Σ-matrix and C' the logic of $\mathcal{B}$. Then, the following are equivalent:*

(i) *C' has a theorem;*
(ii) *L_2 does not form a subalgebra of $\mathfrak{B}^2$;*
(iii) *$\mathcal{B}$ has a semi-conjunction.*

Proof. First, given any semi-conjunction φ of $\mathcal{B}$, $\sim\varphi[x_1/{\sim}x_0]$ is a theorem of C', while $\varphi^{\mathfrak{A}^2}(\langle 0,1\rangle,\langle 1,0\rangle) = \langle 0,0\rangle \notin L_2$, and so (ii) holds. Conversely, given any $\phi \in C'(\varnothing)$, by the structurality of C', $\psi \triangleq (\phi[x_i/x_0]_{i\in\omega}) \in (\mathrm{Fm}^1_\Sigma \cap C'(\varnothing))$, in which case $\sim\psi$ is a semi-conjunction of $\mathcal{B}$. Finally, assume (ii) holds. Then, there are some $\phi \in \mathrm{Fm}^2_\Sigma$ and some $j \in 2$ such that $\phi^{\mathfrak{B}}(i, 1-i) = j$, for all $i \in 2$, in which case $\sim^j\phi$ is a semi-conjunction of $\mathcal{B}$. □

Let $\mathcal{B} \triangleq \mathcal{A}_{\mathrm{PC}}$. Consider the following complementary cases:

- 2 forms a subalgebra of $\mathfrak{A}$,
  in which case $\mathcal{B} = (\mathcal{A}{\restriction}2)$, in view of Corollary 6.3(i), and so binary semi-conjunctions for $\mathcal{A}$ are exactly semi-conjunctions of $\mathcal{B}$, while $L_2 \subseteq 2^2$ forms a subalgebra of $\mathfrak{A}^2$ iff it forms a subalgebra of $\mathfrak{B}^2 = (\mathfrak{A}^2{\restriction}2^2)$.
- 2 does not form a subalgebra of $\mathfrak{A}$.
  Then, by Corollary 6.4(ii), L_4 forms a subalgebra of $\mathfrak{A}^2$, while $\theta^{\mathcal{A}^2{\restriction}L_4} \in \mathrm{Con}(\mathfrak{A}^2{\restriction}L_4)$, in which case $\mathcal{B} = \langle h[\mathfrak{A}^2{\restriction}L_4], \{1\}\rangle$, where $h \triangleq \chi^{\mathfrak{A}^2{\restriction}L_4}$ is a strict surjective homomorphism from $\mathcal{E} \triangleq (\mathcal{A}^2{\restriction}L_4)$ onto $\mathcal{B}$, and so $g : E^2 \to B^2, \langle a,b\rangle \mapsto \langle h(a), h(b)\rangle$ is a surjective homomorphism from $\mathfrak{E}^2$ onto $\mathfrak{B}^2$. In particular, $L_8 \subseteq L_4^2$ forms a subalgebra of $(\mathfrak{A}^2)^2$ iff $L_8 = g^{-1}[L_2]$ forms a subalgebra of $\mathfrak{E}^2 = ((\mathfrak{A}^2)^2{\restriction}L_4^2)$ iff L_2 forms a subalgebra of $\mathfrak{B}^2$. Moreover, $a \triangleq \langle 1, \frac{1}{2}\rangle \in D^{\mathcal{E}} \not\ni b \triangleq \langle 0, \frac{1}{2}\rangle \in E$, in which case we have $h(a|b) \in | \notin D^{\mathcal{B}}$, and so $h(a|b) = (1|0)$. Consider any binary semi-conjunction φ for $\mathcal{A}$. Then, $E \ni \varphi^{\mathfrak{E}}(a|b, b|a) = \varphi^{\mathfrak{A}^2}(a|b, b|a)$, in which case, as $(\pi_0{\restriction}A^2) \in \hom(\mathfrak{A}^2, \mathfrak{A})$, we have $\pi_0(\varphi^{\mathfrak{E}}(a|b, b|a)) = \varphi^{\mathfrak{A}}(\pi_0(a|b), \pi_0(b|a)) = \varphi^{\mathfrak{A}}(1|0, 0|1) = 0$, and so $\varphi^{\mathfrak{E}}(a|b, b|a) \notin D^{\mathcal{E}}$. Hence, $\varphi^{\mathfrak{B}}(1|0, 0|1) = \varphi^{\mathfrak{B}}(h(a|b), h(b|a)) = h(\varphi^{\mathfrak{E}}(a|b, b|a)) \notin D^{\mathcal{B}}$, in which case $\varphi^{\mathfrak{B}}(1|0, 0|1) = 0$, and so φ is a semi-conjunction of $\mathcal{B}$. Conversely, consider any semi-conjunction φ of $\mathcal{B}$. Then, $h(\varphi^{\mathfrak{E}}(a|b, b|a)) = \varphi^{\mathfrak{B}}(h(a|b), h(b|a)) = \varphi^{\mathfrak{B}}(1|0, 0|1) = 0 \notin D^{\mathcal{B}}$, in which case $\langle \varphi^{\mathfrak{A}}(1|0, 0|1), \varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2})\rangle = \varphi^{\mathfrak{E}}(a|b, b|a) \notin D^{\mathcal{E}}$. Consider the following complementary subsubcases:
  - $\varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$.
    Then, as $\frac{1}{2} \in D^{\mathcal{A}}$, $\varphi^{\mathfrak{A}}(1|0, 0|1) = 0$, and so φ is a binary semi-conjunction for $\mathcal{A}$.
  - $\varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2}) \neq \frac{1}{2}$.
    Then, as 2^2 is disjoint with $L_4 = E \ni \varphi^{\mathfrak{E}}(a|b, b|a)$, $\varphi^{\mathfrak{A}}(1|0, 0|1) = \frac{1}{2}$, in which case, as $\frac{1}{2} \in D^{\mathcal{A}}$, $\varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2}) = 0$, and so $\varphi[x_i/\varphi]_{i\in 2}$ is a binary semi-conjunction for $\mathcal{A}$.

In this way, Claim 6.11 completes the argument. □

Theorem 6.12. *Suppose C is $\sim$-subclassical. Then, the following are equivalent:*

(i) *C has a theorem;*
(ii) *C^{PC} has a theorem;*
(iii) *$\mathcal{A}$ has a binary semi-conjunction;*
(iv) *$L_{2[+6]}$ does not form a subalgebra of $(\mathfrak{A}^{[2]})^2$, whenever 2 does [not] form a subalgebra of $\mathfrak{A}$;*
(v) *Any consistent extension of C is a sublogic of C^{PC}.*

Proof. First, (ii)⇔(iii)⇔(iv) are by Lemma 6.10. Next, (i)⇒(ii) is by the inclusion $C(\varnothing) \subseteq C^{\mathrm{PC}}(\varnothing)$. Further, (iii)⇒(i) is by Claim 6.2. Finally, (v)⇒(i) is by Remark 2.1 and the inferential consistency of $\sim$-classical Σ-matrices. Conversely, assume

(iii) holds. Consider any consistent extension C' of C. Then, in case $C' = C$, we have $C' = C \subseteq C^{\mathrm{PC}}$. Now, assume $C' \neq C$, in which case, by (iii) and Theorem 5.1(iii)⇒(i), C' is not $\sim$-paraconsistent, while, by its structurality, $x_0 \notin T \triangleq C'(\varnothing)$, whereas $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C), and so is its consistent finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{1}_{\Sigma}, T \cap \mathrm{Fm}^{1}_{\Sigma}\rangle$. Hence, by Lemma 2.2, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of it, in which case $\mathcal{D}$ is a consistent model of C', for $\mathcal{B}$ is so, and so is non-$\sim$-paraconsistent, for C' is so. Therefore, by (iii), Lemma 6.1 and Theorem 6.4, a Σ-matrix defining C^{PC} is embeddable into $\mathcal{D} \in \mathrm{Mod}(C')$, in which case $C' \subseteq C^{\mathrm{PC}}$, and so (v) holds, as required. □

By Remark 5.3/5.6 and Theorem 6.12(iii/i)⇒(v) /"as well as Lemma 3.3", we then have:

Corollary 6.13. *Suppose C is both $\sim$-subclassical and weakly conjunctive/"disjunctive (in particular, implicative)". Then, any consistent extension of C is a sublogic of C^{PC}.*

On the other hand, the extension $(C^{\mathrm{IC}})_{+0}$ actually invoked in proving (v)⇒(i) of Theorem 6.12 (cf. Remark 2.1) is inferentially-inconsistent. In the next section, we obtain an "inferential" version of this theorem.

6.1.2. *Maximal paraconsistency and quasi-negations versus inferentially-consistent non-subclassical extensions of subclassical 3VLPSN.*

Theorem 6.14. *Suppose C is $\sim$-subclassical. Then, the following are equivalent:*

(i) *any inferentially-consistent extension of C is a sublogic of C^{PC};*
(ii) *$\mathcal{A}$ has a quasi-negation and C is maximally $\sim$-paraconsistent;*
(iii) *$\mathcal{A}$ has a quasi-negation and L_3 does not form a subalgebra of $\mathfrak{A}^2$.*

Proof. First, (ii)⇔(iii) is by Theorem 5.1(i)⇔(iv). Next, by Theorem 5.1(i)⇒(ii), C has a $\sim$-paraconsistent (and so inferentially-consistent) non-$\sim$-subclassical extension, whenever it has a proper $\sim$-paraconsistent extension. Likewise, by the following claim, C has an inferentially-consistent non-$\sim$-subclassical extension, unless $\mathcal{A}$ has a quasi-negation:

Claim 6.15. *Let $\mathcal{B}$ be the submatrix of $\mathcal{A}^2$ generated by $\{\langle 1, \frac{1}{2}\rangle\}$ and C' the logic of $\mathcal{B}$. Suppose $\mathcal{A}$ has no quasi-negation. Then, $(B \setminus D^{\mathcal{B}}) = L_2 \neq \varnothing$, in which case $\sim x_0 \vdash x_0$ is true in $\mathcal{B}$, and so, by (2.11) with $n = 1$ and $m = 0$, $\sim$ is not a subclassical negation for C'. In particular, C' is a non-$\sim$-subclassical inferentially-consistent (for $\mathcal{B}$ is so, because $D^{\mathcal{B}} \ni \langle 1, \frac{1}{2}\rangle$ is neither empty nor equal to B) proper extension of C (cf. Theorem 4.1).*

Proof. Then, $\sim^{\mathfrak{A}} \frac{1}{2} = 1$, while $(B \cap \{0, \frac{1}{2}\}^2) = \varnothing$, in which case $(B \setminus D^{\mathcal{B}}) = L_2 \neq \varnothing$, and so $\sim x_0 \vdash x_0$ is true in $\mathcal{B}$, because, for every $a \in L_2$, $\sim^{\mathfrak{B}} a \in L_2$, as required. □

Thus, (i)⇒(ii) holds. Conversely, assume (ii) holds. Consider any inferentially-consistent extension C' of C. In case $C' = C$, we have $C' = C \subseteq C^{\mathrm{PC}}$. Now, assume $C' \neq C$, in which case C' is not $\sim$-paraconsistent. Then, $x_1 \notin T \triangleq C'(x_0) \ni x_0$. Moreover, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C), and so is its consistent truth-non-empty finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{2}_{\Sigma}, T \cap \mathrm{Fm}^{2}_{\Sigma}\rangle$. Therefore, by Lemma 2.2, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of it, in which case $\mathcal{D}$ is a consistent truth-non-empty model of C', for $\mathcal{B}$ is so, and so $\mathcal{D}$ is non-$\sim$-paraconsistent, for C' is so. Hence, by Lemma 6.1 and Theorem 6.4, a Σ-matrix defining C^{PC} is embeddable into $\mathcal{D} \in \mathrm{Mod}(C')$, in which case $C' \subseteq C^{\mathrm{PC}}$, and so (i) holds, as required. □

Though inferentially-consistent Σ-logics are consistent, Theorem 6.14 is not subsumed by Theorem 6.12, in view of:

Example 6.16. Let $\Sigma \triangleq \{\sim, \neg\}$ with unary $\neg$, $\sim^{\mathfrak{A}}\frac{1}{2} \triangleq 1$ and $\neg^{\mathfrak{A}}a \triangleq (1-a)$, for all $a \in A$. Then, $\neg$ is a quasi-negation for $\mathcal{A}$, while, by Theorem 5.1(iii)$\Rightarrow$(i), C is maximally $\sim$-paraconsistent, whereas 2 forms a subalgebra of $\mathfrak{A}$, in which case, by Theorem 6.4, C is $\sim$-subclassical, and so, by Theorem 6.14(ii)$\Rightarrow$(i), has no inferentially-consistent extension not being a sublogic of C^{PC}. On the other hand, L_2 forms a subalgebra of $\mathfrak{A}^2$. Hence, by Theorem 6.12(i/v)$\Rightarrow$(iv), C has "no theorem"/"a consistent extension not being a sublogic of C^{PC}". ☐

And what is more, in view of the non-optional version of Example 5.9, in which case $\sim$ is a quasi-negation for $\mathcal{A}$, the condition of "maximal $\sim$-paraconsistency"/ "existence of a binary semi-conjunction" as well as any equivalent (in view of "Theorem 5.1(iii)$\Leftrightarrow$(iv)"/"Lemma 6.10") one cannot be omitted in the formulation of Theorem 6.14/6.12, respectively. Likewise, the condition of existence of a quasi-negation cannot be omitted in the formulation of Theorem 6.14, as it ensues from:

Example 6.17. Let $\Sigma \triangleq \Sigma_\sim$ and $\sim^{\mathfrak{A}}\frac{1}{2} \triangleq 1$. Then, C is maximally $\sim$-paraconsistent (cf. Theorem 5.1(iii)$\Rightarrow$(i)), while 2 forms a subalgebra of $\mathfrak{A}$, in which case C is $\sim$-subclassical (cf. Theorem 4.1), whereas $L_2 \cup \{\langle 1, \frac{1}{2}\rangle\}$, being disjoint with $\{0, \frac{1}{2}\}^2 \cup \Delta_A$, forms a subalgebra of $\mathfrak{A}^2$, and so $\mathcal{A}$ has neither any quasi-negation nor any ternary equalizer. ☐

6.2. Disjunctive versus classical extensions of disjunctive 3VLPSN.

Theorem 6.18. *Let C' be an extension of C. Suppose C (viz., $\mathcal{A}$; cf. Lemma 3.2) is $\veebar$-disjunctive. Then, the following are equivalent:*

(i) *C' is $\veebar$-disjunctive, consistent and proper;*
(ii) *C' is $\sim$-classical;*
(iii) *$C' = C^{\mathrm{R}}$ is consistent;*
(iv) *C' is consistent, $\veebar$-disjunctive and not $\sim$-paraconsistent.*

In particular, C is $\sim$-subclassical iff it has a $\veebar$-disjunctive (in particular, axiomatic) consistent proper extension, in which case C^{PC} is a unique one.

Proof. First, (i) is a particular case of (iv). Next, (iii)$\Rightarrow$(iv) is by Theorem 3.5. Further, (iii)$\Rightarrow$(iv) is by Corollaries 6.5 and 6.7.

Now, assume (i) holds. Let us prove, by contradiction, that C is $\sim$-subclassical. For suppose C is not $\sim$-subclassical. Consider the following complementary cases:

- C' is $\sim$-paraconsistent,
  in which case, by Theorem 5.7(i,ii) and (iii)**c)**$\Rightarrow$**d)**, $C' = C_{\frac{1}{2}} \supseteq C$ satisfies (4.2), and so, by the $\veebar$-disjunctivity of C' and Remark 5.6, we have $\sim x_0 \in (C_{\frac{1}{2}}(\sim x_0) \cap C_{\frac{1}{2}}(x_0)) = C_{\frac{1}{2}}(x_0 \veebar \sim x_0) \subseteq C_{\frac{1}{2}}(\varnothing)$. This contradicts to the fact $\sim x_0$ is not true in $\mathcal{A}_{\frac{1}{2}}$ under $[x_0/0]$.
- C' is not $\sim$-paraconsistent,
  in which case, by Lemma 3.4, $C^{\mathrm{R}} \subseteq C'$, and so, by Corollary 6.7, C' is inconsistent. This contradicts to (i).

Thus, in any case, we come to a contradiction, and so C is $\sim$-subclassical. Then, by Corollary 5.9, C' is not $\sim$-paraconsistent, in which case, by Lemma 3.4 and Corollary 6.7, $C^{\mathrm{PC}} \subseteq C'$, and so, by Corollary 6.13, (ii) holds.

Finally, (2.1) and Corollary 6.5 complete the argument. ☐

6.2.1. *Axiomatic versus classical extensions of implicative 3VLPSN.*

Corollary 6.19. *Suppose C (viz., $\mathcal{A}$; cf. Lemma 3.3) is $\sqsupset$-implicative. Then, C is $\sim$-subclassical iff it has an axiomatic consistent proper extension, in which case*

C^{PC} is a unique one and is relatively axiomatized by $\sqsupset \overleftarrow{\langle \bar{\phi}, \phi \rangle}$, where $\bar{\phi} \in (\mathrm{Fm}^1_\Sigma)^*$ and $\psi \in (C^{\mathrm{PC}}(\mathrm{img}\,\bar{\phi}) \setminus C(\mathrm{img}\,\bar{\phi}))$ (in particular, by (2.10)).

Proof. Assume C is $\sim$-subclassical, in which case, by Corollary 6.5, 2 forms a subalgebra of $\mathfrak{A}$, while C^{PC} is defined by $\mathcal{A}{\restriction}2$, and so $\mathbf{S}_*(\mathcal{A}) = \{\mathcal{A}, \mathcal{A}{\restriction}2\}$, while $\sqsubset (\psi, \bar{\phi})$, not being true in the $\sqsupset$-implicative $\mathcal{A}$, is true in the $\sqsupset$-implicative $\mathcal{A}{\restriction}2$. Hence, by Corollary 2.4, C^{PC} is relatively axiomatized by $\sqsubset (\psi, \bar{\phi})$. In this way, Theorem 6.18 completes the argument. □

This subsumes Theorem 6.3 of [12] proved *ad hoc* therein.

7. Structural completeness and completions

Theorem 7.1. *C is structurally complete iff the following hold:*

(i) *C has a theorem;*
(ii) *C is maximally $\sim$-paraconsistent;*
(iii) *$\mathcal{A}$ has a quasi-negation;*
(iv) *$\mathcal{A}$ has a ternary equalizer;*
(v) *L_4 does not form a subalgebra of $\mathfrak{A}^2$;*
(vi) *C is not $\sim$-subclassical,*

in which case any three-valued expansion of C is structurally complete.

Proof. First, assume (i–vi) hold. Consider any extension C' of C axiomatically-equivalent to C. Let us prove, by contradiction, that $C' = C$. For suppose $C' \neq C$, in which case, by (ii), C' is not $\sim$-paraconsistent. Then, by (i), there is some $\varphi \in C(\varnothing)$, in which case, by the consistency of C and the structurality of $C' \equiv_1 C$, $x_0 \notin T \triangleq C'(\varnothing) \ni (\varphi[x_i/x_0]_{i\in\omega}) \in \mathrm{Fm}^1_\Sigma$, while $\langle \mathfrak{Fm}^\omega_\Sigma, T\rangle$ is a model of C' (in particular, of C), and so is its finitely-generated consistent truth-non-empty submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^1_\Sigma, T \cap \mathrm{Fm}^1_\Sigma\rangle$. Hence, by Lemma 2.2, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$, in which case $\mathcal{D}$ is a consistent truth-non-empty model of C', for C is so, and so is non-$\sim$-paraconsistent, for C' is so. This contradicts to (iii–vi), Lemma 6.1(ii) and Theorem 6.4. Thus, $C' = C$, and so C is structurally complete. Conversely, consider the following respective cases:

(i) does not hold, in which case, by Remark 2.1, C is not structurally complete.
(ii) does not hold, in which case, by Theorem 5.7(i), $C_{\frac{1}{2}}$ is a proper extension of C, and so it is axiomatically-equivalent to C, for $\frac{1}{2} \in D^{\mathcal{A}}$ (in particular, C is not structurally complete).
(iii) does not hold, in which case, by Claim 6.15, the logic C' of the submatrix $\mathcal{E}$ of $\mathcal{A}^2$ generated by $\{\langle 1, \frac{1}{2}\rangle\}$ is a proper extension of C, and so it is axiomatically-equivalent to C, for $(\pi_1{\restriction}E) \in \hom^{\mathrm{S}}(\mathcal{E}, \mathcal{A})$, as $A \supseteq \pi_1[E] \supseteq \pi_1[L_2 \cup \{\langle 1, \frac{1}{2}\rangle\}] = (2 \cup \{\frac{1}{2}\}) = A$ (in particular, C is not structurally complete).
(iv) does not hold, in which case the logic C' of the submatrix $\mathcal{E}$ of $\mathcal{A}^2$ generated by $L_2 \cup \{\langle 1, \frac{1}{2}\rangle\}$ is a non-$\sim$-paraconsistent (for $E \not\ni \langle \frac{1}{2}, \frac{1}{2}\rangle \in \Delta_A$, and so proper) extension of C, and so it is axiomatically-equivalent to C, for $(\pi_1{\restriction}E) \in \hom^{\mathrm{S}}(\mathcal{E}, \mathcal{A})$, as $A \supseteq \pi_1[E] \supseteq \pi_1[L_2 \cup \{\langle 1, \frac{1}{2}\rangle\}] = (2 \cup \{\frac{1}{2}\}) = A$ (in particular, C is not structurally complete).
(v) does not hold, the logic C' of $\mathcal{E} \triangleq (\mathcal{A}^2{\restriction}L_4)$ is a non-$\sim$-paraconsistent (for $E = L_4 \not\ni \langle \frac{1}{2}, \frac{1}{2}\rangle \in L_3$, and so proper) extension of C, and so it is axiomatically-equivalent to C, for $(\pi_0{\restriction}E) \in \hom^{\mathrm{S}}(\mathcal{E}, \mathcal{A})$, as $A \supseteq \pi_0[E] \supseteq \pi_0[\{\langle \frac{1}{2}, 1\rangle, \langle 0, \frac{1}{2}\rangle, \langle 1, \frac{1}{2}\rangle\}] = A$ (in particular, C is not structurally complete).
(vi) does not hold, in which case, by Theorem 6.8(ii)⇒(iv), C^{NP} is a proper extension of C, axiomatically-equivalent to this, and so C is not structurally complete.

Finally, as expansions of $\mathcal{A}/C$ inherit (iii-v)/"both (i) and absence of $\sim$-classical models", respectively, Corollary 4.6 and the last assertion of Theorem 5.1 complete the argument. □

Remark 7.2. Let φ be a binary semi-conjunction for $\mathcal{A}$. Then, it is a ternary equalizer for $\mathcal{A}$, while, in case $\sim^{\mathfrak{A}}\frac{1}{2} = (\frac{1}{2}/1)$, $\sim/(\varphi[x_i/\sim^{i+1}x_0]_{i\in 2})$ is a quasi-negation for $\mathcal{A}$. □

Remark 7.3. Suppose $\mathcal{A}$ is weakly $\overline{\wedge}$-conjunctive. Then, $(\langle 0, \frac{1}{2}\rangle \overline{\wedge}^{\mathfrak{A}^2} \langle \frac{1}{2}, 0\rangle) = \langle 0, 0\rangle \notin L_4 \supseteq \{\langle 0, \frac{1}{2}\rangle, \langle \frac{1}{2}, 0\rangle\}$. Hence, L_4 does not form a subalgebra of $\mathfrak{A}^2$. □

In this way, by Remarks 5.3, 7.2, 7.3, Claim 6.2 and Theorem 7.1, we have:

Corollary 7.4. *[Providing $\mathcal{A}$ (viz., C) is weakly conjunctive] C is structurally-complete [if and] only if it is not $\sim$-classical.*

Combining Lemma 3.3 and Remark 4.2 with Corollaries 5.4, 6.5 and 7.4, we then have:

Corollary 7.5. *Let $c \notin \Sigma$ be a nullary connective, $\Sigma' \triangleq (\Sigma \cup \{c\})$, $\mathcal{A}'$ the Σ'-expansion of $\mathcal{A}$ with $c^{\mathfrak{A}'} \triangleq \frac{1}{2}$ and C' the logic of $\mathcal{A}'$. Suppose $\mathcal{A}$ is both classically-hereditary and weakly conjunctive (in particular, both classically-valued and implicative [i.e., disjunctive]). Then, C' is structurally complete, while C is not so, whereas both C and C' are maximally $\sim$-paraconsistent.*

This covers, in particular, both LP, $\mathbb{S}_3$, HZ, $PG3^{[*]}$ (as non-classically-valued conjunctive classically-hereditary instances) and P^1 (as a classically-valued implicative instance) as well as their bounded expansions by solely nullary connectives taking the classical values 0 and 1. (In this connection, recall that the fact that LP is "maximally $\sim$-paraconsistent"/"not structurally complete" has been due to [13]/[15], respectively, proved *ad hoc* therein.) Thus, in view of Theorems 4.1, 7.1 and Corollary 7.5, any $\sim$-paraconsistent three-valued Σ-logic with subclassical negation $\sim$ is maximally so, whenever it is structurally complete, while the converse does not, generally speaking, hold, whereas the structural completeness of such a logic subsumes absence of its $\sim$-classical extensions. On the other hand, Lemma 3.3, Theorem 4.1, Corollaries 6.5, 7.4, 7.5 and Remark 4.2 as well as the mentioned instances inevitably raise the problem of finding the structural completions of conjunctive $\sim$-subclassical [viz., classically-hereditary (in particular, both classically-valued and implicative/disjunctive/conjunctive)] three-valued $\sim$-paraconsistent Σ-logics. We start from providing an effective algebraic criterion of the $\sim$-classicism of their structural completions (in the next subsection) and then (in the next two sections) solve the problem as such within two disjoint generic contexts covering, in particular, the above instances.

7.1. **Classicism of the structural completions of genuinely subclassical 3VLPSN.** A *[classical false-relative] binary negation for/of $\mathcal{A}$* is any $\varphi \in \mathrm{Fm}^2_\Sigma$ such that, for all $a \in A$, $\varphi^{\mathfrak{A}}(a, 0) = (1 - \chi^{\mathcal{A}}(a))$ (in particular, when both $\mathcal{A}$ is classically-hereditary and either $\mathcal{A}$ is φ-implicative or both $\varphi \in \mathrm{Fm}^1_\Sigma$ and $\mathcal{A}$ is φ-negative).

Lemma 7.6. *$\mathcal{A}$ has a binary negation iff $\langle 1, 0, 0\rangle \in K'_2$. In particular, providing $\mathcal{A}$ is classically-hereditary, the following are equivalent:*

(i) *$\mathcal{A}$ is implicative;*
(ii) *$\mathcal{A}$ is disjunctive and has a binary negation;*
(iii) *$\mathcal{A}$ is disjunctive, while $\langle 1, 0, 0\rangle \in K'_2$.*

Proof. The first assertion (in particular, (ii)⇔(iii)) as well as (i)⇒(ii) are immediate. Conversely, if $\mathcal{A}$ is $\veebar$-disjunctive and φ is a binary negation for it, then it is $(\varphi \veebar x_1)$-implicative. □

Lemma 7.7. *[Providing $\mathcal{A}$ is classically-hereditary] it holds that (i)⇔(ii)⇔(iii)⇔(iv)⇒(v)⇔(vi)[⇒(i)], where:*

(i) $\langle 0,1\rangle \in K'_3$;
(ii) $\langle 1,0\rangle \in K'_3$;
(iii) $L_2 \subseteq K'_3$;
(iv) $(L_2 \cap K'_3) \neq \varnothing$;
(v) $K'_3 \nsubseteq K_4 \triangleq (K_3 \cup \{\langle \frac{1}{2}, 1\rangle\})$;
(vi) *neither* K_3 *nor* K_4 *forms a subalgebra of* $\mathfrak{A}^2$.

Proof. First, (i)⇔(ii) is by the fact that $\sim^{\mathfrak{A}} j = (1-j)$, for all $j \in 2$, while (iii/iv) is the conjunction/disjunction of (i) and (ii), respectively, and so equivalent to these. Next, (iii)⇒(v) is by the fact that $L_2 \nsubseteq K_4$. Further, (v)⇒(vi) is by the inclusion $K_3 \subseteq K_4$. The converse is by the fact that $(K_4 \setminus K_3) = \{\langle \frac{1}{2}, 1\rangle\}$ is a singleton, while $K_3 \subseteq K'_3$. [Finally, (v)⇒(iv) is by the fact that $K_4 = ((A \times 2) \setminus L_2)$, while $K'_3 \subseteq (A \times 2)$, for $\pi_1[K_3] = 2$ forms a subalgebra of $\mathfrak{A}$]. □

Theorem 7.8. *Suppose C is $\sim$-subclassical. Let:*

(i) C^{PC} *is a proper axiomatic extension of* C;
(ii) $C^{\mathrm{PC}} \not\equiv_1 C$;
(iii) C^{PC} *is not the structural completion of* C;
(iv) *the structural completion of C is not $\sim$-classical;*
(v) $\langle 0,1\rangle \in K'_3$;
(vi) *neither* K_3 *nor* K_4 *forms a subalgebra of* $\mathfrak{A}^2$;
(vii) $\mathcal{A}$ *has a binary negation;*
(viii) C *is implicative.*

Then, the following hold:

a) *it holds that (vi)⇐(v)⇐(vii)⇐(viii)⇒(i)⇒(ii) ⇒(iii)⇔(iv);*
b) *providing C^{PC} has a theorem, it holds that (iii)⇒(ii);*
c) *providing C^{PC} is $\veebar$-disjunctive (in particular, C is so; cf. Lemma 3.2), (i–vi) are equivalent to one another;*
d) *providing C is $\barwedge$-conjunctive, (i–vii) are equivalent to one another;*
e) *providing C is both conjunctive and disjunctive, (i–viii) are equivalent to one another.*

Proof. **a)** First, (ii/iii) is a particular case of (i/(ii|iv)), respectively, while (iii)⇒(iv) is by Theorem 6.4. Next, (viii)⇒(i/vii) is by "Corollary 6.19"/"Lemma 3.3", respectively. Further, (v)⇒(vi) is by Lemma 7.7(i)⇒(vi). Finally, if φ is a binary negation for $\mathcal{A}$, then $K'_3 \ni \varphi^{\mathfrak{A}^2}(\langle \frac{1}{2}, 0\rangle, \langle 0,0\rangle) = \langle 0,1\rangle$, for $K'_3 \supseteq K_3 \supseteq \{\langle \frac{1}{2}, 0\rangle, \langle 0,0\rangle\}$ forms a subalgebra of $\mathfrak{A}^2$, in which case (v) holds, and so does (vii)⇒(v), as required.

b) is by Theorem 6.12(ii)⇒(v) and the consistency of $\sim$-classical Σ-logics, and so of their axiomatically-equivalent extensions.

c) In that case, by the optional version of Theorem 6.4, 2 forms a subalgebra of $\mathfrak{A}$, $\mathcal{A}{\upharpoonright}2$ being $\veebar$-disjunctive and defining C^{PC}. First, assume (ii) holds. Take any $\varphi \in (C^{\mathrm{PC}}(\varnothing) \setminus C(\varnothing)) \neq \varnothing$, for $C \subseteq C^{\mathrm{PC}}$, in which case it is true in $\mathcal{A}{\upharpoonright}2$ but not true in $\mathcal{A}$, and so, by Corollary 2.4, C^{PC} is the axiomatic extension of C relatively axiomatized by φ, for $\mathbf{S}_*(\mathcal{A}) = \{\mathcal{A}, \mathcal{A}{\upharpoonright}2\}$. Thus, (i) holds. Furthermore, if $\langle 0,1\rangle \notin K'_3$, then $(\pi_1{\upharpoonright}K'_3) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{K}'_3, \mathcal{A}{\upharpoonright}2)$, in which case $C^{\mathrm{PC}} \supseteq C$ is equally defined by $\mathcal{K}'_3$, and so is axiomatically-equivalent to C, for $(\pi_0{\restriction}K'_3) \in \mathrm{hom}^{\mathrm{S}}(\mathcal{K}'_3, \mathcal{A})$. Hence, (ii)⇒(v) holds. Conversely, assume (v) holds. Then, there is some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2}/0, 0, 1) = (0/1)$. Moreover, as 2 forms a subalgebra of $\mathfrak{A}$, $\varphi^{\mathfrak{A}}(1, 0, 1) \in 2$. Consider the following complementary cases:

- $\varphi^{\mathfrak{A}}(1, 0, 1) = 1$,
  in which case $\varphi[x_{2-i}/{\sim}^i(x_1 \veebar {\sim}x_1)]_{i\in 2}$ is true in $\mathcal{A}{\restriction}2$ but not true in $\mathcal{A}$ under $[x_0/\frac{1}{2}, x_1/0]$, and so (ii) holds.
- $\varphi^{\mathfrak{A}}(1, 0, 1) = 0$,
  in which case $\veebar\langle\varphi[x_0/{\sim}^j x_0; x_{2-i}/{\sim}^i(x_1 \veebar {\sim}x_1)]_{i\in 2}\rangle_{j\in 2}$ is true in $\mathcal{A}{\restriction}2$ but not true in $\mathcal{A}$ under $[x_0/\frac{1}{2}, x_1/0]$, and so (ii) holds.

Thus, in any case, (ii) holds. In this way, **a)**, **b)**, Remark 5.6 and the optional version of Lemma 7.7(vi)⇒(i) complete the argument.

d) In that case, $C^{\mathrm{PC}} \supseteq C$ is $\bar{\wedge}$-conjunctive, and so $\bar{\wedge}^{\sim}$-disjunctive, for $\mathcal{A}_{\mathrm{PC}}$, defining it, is both $\sim$-negative and then $\bar{\wedge}$-conjunctive. Therefore, **c)** with $\veebar = \bar{\wedge}^{\sim}$ holds. Furthermore, if (v) holds, then there is some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2}/0, 0, 1) = (0/1)$, in which case $(\varphi[x_{1+i}/{\sim}^i(x_1 \bar{\wedge} {\sim}x_1)]_{i\in 2}) \bar{\wedge} {\sim}x_0$ is a binary negation for $\mathcal{A}$, and so (vii) holds. In this way, **a)**(vii)⇒(v) completes the argument.

e) is by **a)**(viii)⇒(vii), **d)** and Lemmas 3.2 and 7.6(ii)⇒(i). □

The condition of conjunctivity is essential for Theorem 7.8**d)**,**e)** to hold, by:

Example 7.9. Let $\Sigma \triangleq (\Sigma^{\supset}_{\sim,+} \setminus \{\wedge\})$, ${\sim}^{\mathfrak{A}}\frac{1}{2} \triangleq \frac{1}{2}$ and

$$(a(\vee|\supset)^{\mathfrak{A}}b) \triangleq \begin{cases} \frac{1}{2}|0 & \text{if } \frac{1}{2} \in \{a, b\}, \\ \max(a, b)|1 & \text{otherwise,} \end{cases}$$

for all $a, b \in A$. Then, 2 forms a subalgebra of $\mathfrak{A}$, in which case, by Theorem 6.4, C is $\sim$-subclassical, C^{PC} being defined by $\mathcal{A}{\restriction}2$, while $\mathcal{A}$ is $\vee$-disjunctive, and so is C, whereas $\langle 0, 1\rangle = (\langle\frac{1}{2}, 0\rangle \supset^{\mathfrak{A}^2} \langle 0, 0\rangle) \in K'_3$, for $K'_3 \supseteq K_3 \supseteq \{\langle\frac{1}{2}, 0\rangle, \langle 0, 0\rangle\}$ forms a subalgebra of $\mathfrak{A}^2$, in which case 7.8(v) holds, and so do 7.8(i–iv,vi), in view of Theorem 7.8**c)**. On the other hand, $((2^2 \times \{\frac{1}{2}\}) \cup (\Delta_2 \times 2)) \supseteq K_2$ forms a subalgebra of $\mathfrak{A}^3$ but does not contain $\langle 1, 0, 0\rangle$, for $1 \neq 0 \neq \frac{1}{2}$, in which case, by Lemma 7.6(i/ii)⇒(iii), $\mathcal{A}$ "is not implicative"/"has no binary negation, and so is not conjunctive, in view of Theorem 7.8**d)**(v)⇒(vii)". □

In general, Theorem 7.8**e)**(v[i])⇔(viii) yields a quite useful effective algebraic criterion of the implicativity of disjunctive conjunctive $\sim$-subclassical three-valud $\sim$-paraconsistent Σ-logics. For example, when $C = LP$ (cf. Subsubsection 4.1.1), $K_{4[-1]}$ does [not] form a subalgebra of $\mathfrak{A}^2$, so C is not implicative. Likewise, when $C = PG3^{\{*\}}$ (cf. Paragraph 4.1.4.2 {resp., 4.1.4.1}), $K_{3[+1]}$ does [not] form a subalgebra of $\mathfrak{A}^2$, so C is not implicative (just like its prototype — {the implication-less fragment of} Gödel three-valued logic [3]). In particular, these instances show that both K_3 and K_4 are essential within the item (vi) of both Lemma 7.7 and Theorem 7.8. And what is more, we have the following universal observation, covering arbitrary disjunctive conjunctive $\sim$-subclassical three-valued $\sim$-paraconsistent Σ-logics with lattice conjunction and disjunction but with, so to say, "bizarre" lattice partial ordering (in particular, $[NI]HZ$, and so providing a generic insight into its implicativity; cf. Subsubsection 4.1.2 [resp., Paragraph 4.1.2.1]):

Corollary 7.10. *Suppose C is both $\bar{\wedge}$-conjunctive and $\veebar$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 3.2) as well as $\sim$-subclassical (i.e., $\mathcal{A}$ is classically-hereditary; cf. Corollary 6.5), while $\mathfrak{A}$ is a $(\bar{\wedge}, \veebar)$-lattice with unit $\frac{1}{2}$ (and so zero 0, for $\mathcal{A}$ is both false-singular with non-distinguished value 0 and $\bar{\wedge}$-conjunctive). Then, C is implicative (viz., $\mathcal{A}$ is so; cf. Lemma 3.3).*

Proof. In that case, $K'_3 \ni \sim^{\mathfrak{A}^2}(\langle\frac{1}{2},0\rangle \mathbin{\overline{\wedge}}^{\mathfrak{A}^2} \langle 1,1\rangle) = \langle 0,1\rangle$, for $K'_3 \supseteq K_3 \supseteq \{\langle\frac{1}{2},0\rangle, \langle 1,1\rangle\}$ forms a subalgebra of $\mathfrak{A}^2$. In this way, Theorem 7.8**e)**(v)⇒(viii) completes the argument. □

8. Extensions

By C^{DMP} we denote the extension of C relatively axiomatized by the *Dual Modens Ponens* rule:

$$\{\sim x_0, x_0 \mathbin{\underline{\vee}} x_1\} \vdash x_1, \tag{8.1}$$

being actually dual to (3.2), in which case C^{DMP} is an/a extension/sublogic of $C^{\mathrm{NP/R}}$, whenever C (viz., $\mathcal{A}$) is weakly $\underline{\vee}$-disjunctive. Likewise, by C^{DN} we denote the extension of C relatively axiomatized by the *Double Negation* rule:

$$x_0 \vdash \sim\sim x_0. \tag{8.2}$$

Remark 8.1. Providing $(2|A)\|(L|K)_{5|(5/6)}$ forms a subalgebra of $\mathfrak{A}^{\|2}$, (8.2) is true in $((\mathcal{A}{\restriction}2)|\mathcal{A})\|(\mathcal{L}|\mathcal{K})_{5|(5/6)}$ |"iff $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$", respectively. □

Let $n \in (\omega\backslash 1)$, C_n the finitary (for C, being three-valued, is so) extension of C relatively axiomatized by the finitary Σ-rule $R_n \triangleq ((\{\sim x_i \mid i \in n\} \cup \{\underline{\vee}\bar{x}_n\}) \vdash x_n)$ and C_ω the finitary (for C, being three-valued, is so) extension of C relatively axiomatized by the finitary Σ-calculus $\{R_m \mid m \in (\omega \setminus 1)\}$, in which case $\mathrm{Mod}^{[\mathfrak{Fm}^{\omega}_{\Sigma}]}(C_\omega) = (\bigcap_{m\in(\omega\setminus 1)} \mathrm{Mod}^{[\mathfrak{Fm}^{\omega}_{\Sigma}]}(C_m))$ [and so $(\mathrm{img}\, C_\omega) = (\bigcap_{m\in(\omega\setminus 1)}(\mathrm{img}\, C_m))$]. (Note that $R_1 = (2.9)$. In particular, $C^{\mathrm{NP}} = C_1 \subseteq C_\omega$.)

Remark 8.2. Suppose C is weakly $\underline{\vee}$-disjunctive. Then, for all $n \in (\omega \setminus 1)$, by the structurality of C_{n+1}, $R_n[x_{n+1}/x_n, x_n/x_0]$ is satisfied in C_{n+1}, and so is R_n, for C_{n+1}, being an extension of C, is weakly $\underline{\vee}$-disjunctive, in which case $C_n \subseteq C_{n+1}$, and so $\{C_m \mid m \in (\omega \setminus 1)\}$ is a chain (w.r.t. $\subseteq$) of finitary Σ-logics, the point-wise union of which is its join in the complete lattice (w.r.t. $\subseteq$) of all Σ-logics that is equal to C_ω. In particular, any Σ-rule is satisfied in C_ω iff, for some $n \in (\omega \setminus 1)$, it is satisfied in C_n. □

Let $\nabla^{\wr}_{\diamond} \triangleq \{\wr x_0 \approx (x_0 \diamond \wr x_0)\} \subseteq \mathrm{Eq}^1_\Sigma$.

Lemma 8.3. *The following hold:*

(i) *providing $\mathcal{A}$ is [both either weakly conjunctive or truth-symmetric and] strongly $\underline{\vee}$-disjunctive, it holds that (B)⇔(C)⇒(A)[⇒(B)], where:*
 (A) *C^{DMP} is a proper extension of C^{NP};*
 (B) *C is $\sim$-subclassical;*
 (C) *K_6 is $\mathfrak{A}^2$-closed, while $\mathcal{K}_6 \in (\mathrm{Mod}(C_\omega) \setminus \mathrm{Mod}(C^{\mathrm{DMP}}))$;*

(ii) *providing $\mathcal{A}$ is both $\underline{\vee}$-disjunctive and $\sqsupset$-implicative (viz., C is so; cf. Lemmas 3.2 and 3.3), C^{DMP} is the extension of C relatively axiomatized by:*

$$\sim x_0 \vdash (x_0 \sqsupset x_1); \tag{8.3}$$

(iii) *providing $\mathcal{A}$ is [both $\underline{\vee}$-disjunctive and] $\overline{\wedge}$-conjunctive (viz., $\mathcal{A}$ is so [cf. Lemma 3.2]), while K_5 forms a subalgebra of $\mathfrak{A}^2$, in which case $\mathfrak{A}$ is classically-hereditary, and so C is $\sim$-subclassical, C^{PC} being defined by $\mathcal{A}{\restriction}2$ (cf. Theorem 6.4),* **a)**, **b)**, **d)** *and* **g)** *[as well as both* **c)**, **e)**, **f)** *and* **h)**–**k)***] hold, where:*
 a) *$\sim^{\mathfrak{A}}\frac{1}{2} = 1 = (\frac{1}{2} \mathbin{\overline{\wedge}}^{\mathfrak{A}} 1) = (1 \mathbin{\overline{\wedge}}^{\mathfrak{A}} \frac{1}{2})$ (in particular, $(\frac{1}{2} \mathbin{\underline{\vee}}^{\mathfrak{A}} 1) = \frac{1}{2}$, whenever $\mathfrak{A}$ is a $(\overline{\wedge}, \underline{\vee})$-lattice);*
 b) *$\mathcal{A}$ is generated by $\{\frac{1}{2}\}$;*
 c) *$\mathcal{K}_5$ is generated by $K_5 \setminus 2^2$;*
 d) *$\mathcal{A}$ is $\neg^{\sim}_{\overline{\wedge}}$-negative [and so $\sqsupset^{\sim}_{\overline{\wedge},\underline{\vee}}$-implicative];*

e) C^{DMP} *is the extension of* C *relatively axiomatized by:*

$$\sim x_0 \vdash (x_0 \sqsupset^{\sim}_{\bar\wedge,\veebar} \sim\sim x_0); \tag{8.4}$$

f) $\mathcal{K}_5 \in \mathrm{Mod}(C^{\mathrm{DMP}})$;

g) $\nabla^{\neg\tilde{\bar\wedge}}_{\bar\wedge}$ *defines truth in* $\mathcal{A}$;

h) $\nabla^{\neg\tilde{\bar\wedge}}_{\bar\wedge}[x_0/(\varepsilon^{\sim}_{\sqsupset^{\sim}_{\bar\wedge,\veebar},\bar\wedge}\sqsupset^{\sim}_{\bar\wedge,\veebar}(\varepsilon^{\sim}_{\sqsupset^{\sim}_{\bar\wedge,\veebar},\bar\wedge}[x_i/x_{2+i}]_{i\in 2}))]$ *is an implicative system for* $\mathfrak{A}$;

i) *the logic of* $\mathcal{K}_5$ *is:*
  (1) *a proper sublogic of* C^{PC};
  (2) *a proper extension of* C^{DMP};

j) *providing* $\mathfrak{A}$ *is a* $(\bar\wedge,\veebar)$*-lattice,* $\mathcal{K}_6$ *is generated by* $K_6 \setminus K_5$*, in which case* $L_5 \subsetneq (\supseteq)(K_6(\setminus K_5))$ *does not form a subalgebra of* $\mathfrak{A}^2$*, and so* $\mathcal{A}$ *is not classically-valued;*

k) *Let* $\mathbb{Q}$ *be the quasivariety generated by* $\mathfrak{A}$ *and* $\mathsf{S} \triangleq \{\mathfrak{A}, \mathfrak{A}{\restriction}2\}$*, in which case* S *is the class of all non-one-element subalgebras of* $\mathfrak{A}$*, is a* skeleton *(i.e., contains no distinct isomorphic members), has no member having a one-element subalgebra, in view of* **b)***, has an implicative system, in view of* **h)***, and generates the quasi-variety* $\mathbb{Q}$*, and so satisfies the defining condition of the first paragraph of Section 4 of* [19] *as for both* $U_{\mathbb{S}} = \varnothing$ *and the implicative* finitely-generated *(viz., generated by a finite class of finite* Σ*-algebras, and so being* locally-finite *[i.e., having no infinite finitely-generated member]) quasi-variety* $\mathbb{Q}$*, in view of Corollary 2.9(2) therein. Then, the following hold:*
  (a) $\varepsilon^{\sim}_{\sqsupset^{\sim}_{\bar\wedge,\veebar},\bar\wedge}$ *is an axiomatic canonical equality determinant for* $\mathcal{A}$ *(in particular,* $\mathbb{Q}$ *is equivalent to* C *with respect to* $\nabla^{\neg\tilde{\bar\wedge}}_{\bar\wedge}$ *and* $\varepsilon^{\sim}_{\sqsupset^{\sim}_{\bar\wedge,\veebar},\bar\wedge}$ *in the sense of* [14]*);*
  (b) $\{$*providing* $\mathfrak{A}$ *is a* $(\bar\wedge,\veebar)$*-lattice*$\}$ $\mathbb{Q}$ *is a variety* $\{$*if and*$\}$ *only if* $(\ker \chi^2_A) \notin \mathrm{Con}(\mathfrak{A})$ *(in particular,* $\mathbb{Q}$ *is not a variety, whenever* $\mathcal{A}$ *is classically-valued);*
  (c) $\boldsymbol{\tau}^{\mathbb{Q}}_{\mathsf{S}}$ *(cf.* [19]*) is not injective (in particular,* S *does not satisfy the condition of Corollary 4.13 of* [19]*, while* $\mathfrak{A}$ *has no congruence-permutation term, and so no discriminator).*

Proof. (i) Then, by the $\veebar$-disjunctivity of C, $C^{\mathrm{NP}} \subseteq C^{\mathrm{DMP}}$. First, (C)⇒(B) is by Theorem 4.1. Next, (C)⇒(A) is by the fact that $C^{\mathrm{NP}} = C_1 \subseteq C_\omega$, due to which $\mathrm{Mod}(C_\omega) \subseteq \mathrm{Mod}(C^{\mathrm{NP}})$. Further, if (B) holds, then, by Corollary 6.5, $\mathcal{A}$ is classically-hereditary, in which case K_6 forms a subalgebra of $\mathfrak{A}^2$, and so $\mathcal{K}_6$ is a model of C, in which (8.1) is not true under $[x_0/\langle\frac{1}{2},0\rangle, x_1/\langle 0,1\rangle]$, for $\mathcal{A}$ is $\veebar$-disjunctive, while $\sim^{\mathfrak{A}}\frac{1}{2} \in D^{\mathcal{A}}$, whereas, since $S \triangleq \{a \in K_6 \mid \sim^{\mathfrak{K}_6} a \in D^{\mathcal{K}_6}\} \subseteq (A \times \{0\})$, while $\mathcal{A}$ is $\veebar$-disjunctive, whereas $(A \setminus D^{\mathcal{A}}) = \{0\}$, for all $n \in (\omega \setminus 1)$ and any $\bar b \in S^n$, we have $\pi_1(\veebar^{\mathfrak{K}_6}\bar b) = 0$, in which case we get $(\veebar^{\mathfrak{K}_6}\bar b) \notin D^{\mathcal{K}_6}$, and so R_n is true in $\mathcal{K}_6$ (in particular, (C) holds). [Otherwise, by the optional version of Theorem 6.8(i)⇒(ii), C^{NP} is inconsistent, and so is its extension C^{DMP}, in which case they are equal. Thus, (A)⇒(B) holds.]

(ii) Since $\mathcal{A}$ is both $\veebar$-disjunctive and $\sqsupset$-implicative, the axiom $x_0 \veebar (x_0 \sqsupset x_1)$ and the rule $\{x_0 \sqsupset x_1, x_0 \veebar x_1\} \vdash x_1$ are true in it, that is, satisfied in C, and so in its extensions, in which case any extension of C satisfies (8.1) iff it satisfies (8.3).

(iii) **a)** If it held that $(\sim^{\mathfrak{A}}\frac{1}{2} \neq 1)|(((\frac{1}{2}/1)\bar\wedge^{\mathfrak{A}}(1/e)) \neq 1)$, then, by the "fact that $\sim^{\mathfrak{A}}\frac{1}{2} \neq 0$"|"$\bar\wedge$-conjunctivity of $\mathcal{A}$", we would have $(\sim^{\mathfrak{A}^2}\langle\frac{1}{2},1\rangle|(\langle\frac{1}{2}/1, 1/0\rangle \bar\wedge^{\mathfrak{A}^2} \langle 1/\frac{1}{2}, 0/1\rangle)) = \langle\frac{1}{2},0\rangle \notin K_5$, in which case $K_5 \supseteq \{\langle\frac{1}{2},1\rangle, \langle 1,0\rangle\}$

would not form a subalgebra of $\mathfrak{A}^2$. (Then, providing $\mathfrak{A}$ is a $(\bar{\wedge},\underline{\vee})$-lattice, by one of the dual absorption lattice identities, we get $(\frac{1}{2}\underline{\vee}^{\mathfrak{A}}1) = (\frac{1}{2}\underline{\vee}^{\mathfrak{A}}(\frac{1}{2}\bar{\wedge}^{\mathfrak{A}}1)) = \frac{1}{2}$.)

b) Then, by **a)**, we have $(\sim^{\mathfrak{A}})^{2-i}\frac{1}{2} = i$, for all $i \in 2$.

c) Likewise, by **a)**, we have $(\sim^{\mathfrak{A}^2})^{2-i}\langle\frac{1}{2},1\rangle = \langle i, 1-i\rangle$, for all $i \in 2$, while, by the $\underline{\vee}$-disjunctivity of $\mathcal{A}{\restriction}2$, $(\langle 0,1\rangle \underline{\vee}^{\mathfrak{A}^2} \langle 1,0\rangle) = \langle 1,1\rangle$, whereas, by the $\bar{\wedge}$-conjunctivity of $\mathcal{A}$, $(\langle 0,1\rangle \bar{\wedge}^{\mathfrak{A}^2} \langle 1,0\rangle) = \langle 0,0\rangle$.

d) is by **a)** and the $\bar{\wedge}$-conjunctivity of $\mathcal{A}{\restriction}2$ [as well as the $\underline{\vee}$-disjunctivity of $\mathcal{A}$].

e) In that case, by **d)**, $\mathcal{A}$ is $\sqsupset$-implicative with $\sqsupset \triangleq \sqsupset^{\sim}_{\bar{\wedge},\underline{\vee}}$, and so, by (ii), C^{DMP} is the extension of C relatively axiomatized by (8.3). On the other hand, (8.4) = (8.3)$[x_1/{\sim}{\sim}x_0]$. Therefore, any extension of C, being structural, satisfies (8.4), whenever it satisfies (8.3). Conversely, by **a)** and the $\sqsupset$-implicativity of $\mathcal{A}$, the rule $\{\sim x_0, x_0 \sqsupset {\sim}{\sim}x_0\} \vdash (x_0 \sqsupset x_1)$ is true in $\mathcal{A}$, that is, satisfied in C, and so in its extensions, in which case any extension of C satisfies (8.3), whenever it satisfies (8.4).

f) Then, by Corollary 6.7, (8.1), being satisfied in C^{R}, is true in $(\mathcal{A}{\restriction}2)^2 = (\mathcal{K}_5{\restriction}2^2)$, and so is (8.4), in view of **e)**, while $(K_5 \setminus 2^2) = \{\langle\frac{1}{2},1\rangle\}$, whereas $\sim^{\mathfrak{K}_5}\langle\frac{1}{2},1\rangle = \langle\sim^{\mathfrak{A}}\frac{1}{2},0\rangle \notin D^{\mathcal{K}_5}$, in which case (8.4) is true in $\mathcal{K}_5$ under $[x_0/\langle\frac{1}{2},1\rangle]$, and so is true in $\mathcal{K}_5 \in \mathrm{Mod}(C)$, as required, in view of **e)**.

g) is by **d)** and the $\bar{\wedge}$-conjunctivity of $\mathcal{A}$.

h) Then, by Remark 4.3, **d)** and the $\bar{\wedge}$-conjunctivity of $\mathcal{A}$, $\varepsilon^{\sim}_{\sqsupset^{\sim}_{\bar{\wedge},\underline{\vee}},\bar{\wedge}}$ is an axiomatic canonical equality determinant for $\mathcal{A}$, and so **d)** and **g)** complete the argument.

i) (1) is by **d)**, Corollary 6.13, Theorem 7.8**a)**(viii)⇒(ii) and the consistency of $\mathcal{K}_5$ as well as the axiomatic equivalence of its logic to C.

(2) We start from proving the following two claims:

Claim 8.4. *Suppose $\mathcal{A}$ is both conjunctive and disjunctive, while K_5 forms a subalgebra of $\mathfrak{A}^2$, in which case $\mathcal{A}$ is classically-hereditery, and so K_6 forms a subalgebra of $\mathfrak{A}^2$. Let I be a set, $\Bbbk_I \triangleq \{a \in K_6^I \mid (\exists i \in I : \pi_i(a) \notin K_5) \Rightarrow (\exists j \in 2 : \pi_j(\pi_i(a)) = 1)\}$ and $\mathcal{B}$ a submatrix of $\mathcal{K}_6^I$. Then, the following hold:*

(i) *$(\mathcal{B} \in \mathrm{Mod}(C^{\mathrm{DMP}})) \Leftrightarrow ((B \setminus K_5^I) \subseteq \Bbbk_I)$;*

(ii) *providing $\mathcal{B}$ is generated by any (non-empty, unless Σ contains a nullary connective) $S \subseteq K_6^I$ such that $(\{|I|,|S|\} \cap 2) \neq \varnothing$, $(B \setminus K_5^I) \subseteq S$;*

(iii) *$\mathcal{B}$ is a model of C^{DMP}, whenever it is generated by any (non-empty, unless Σ contains a nullary connective) $S \subseteq \Bbbk_I$ such that $(\{|I|,|S|\} \cap 2) \neq \varnothing$.*

Proof. (i) First, assume $(B \setminus K_5^I) \subseteq \Bbbk_I$. Consider any $b \in B \subseteq K_6^I$. Then, in case $b \in K_5^I$, by Lemma 8.3(iii)**e)**,**f)**, (8.4) is true in $\mathcal{B}$ under $[x_0/b]$. Otherwise, $b \in (\Bbbk_I \setminus K_5^I)$, in which case there are some $i \in I$ and some $j \in 2$ such that $\pi_j(\pi_i(b)) = 1$, and so $\pi_j(\pi_i(\sim^{\mathfrak{B}}b)) = \sim^{\mathfrak{A}}1 = 0 \notin D^{\mathcal{A}}$, in which case $\sim^{\mathfrak{B}}b \notin D^{\mathcal{B}}$, and so (8.4) is true in $\mathcal{B}$ under $[x_0/b]$. Thus, (8.4) is true in $\mathcal{B} \in \mathrm{Mod}(C)$, and so, by Lemma 8.3(iii)**e)**, $\mathcal{B} \in \mathrm{Mod}(C^{\mathrm{DMP}})$.

Conversely, assume $(B\setminus K_5^I) \nsubseteq \Bbbk_I$. Take any $a \in ((B\setminus K_5^I)\setminus \Bbbk_I) \neq \varnothing$, in which case there is some $i \in I$ such that $\pi_i(a) = \langle \frac{1}{2}, 0\rangle$, while, for all $j \in I$ and all $k \in 2$, $\pi_k(\pi_j(a)) \neq 1$, and so, by Lemma 8.3(iii)**a)**, $\pi_k(\pi_j(\sim^{\mathfrak{B}} a)) = 1 \in D^{\mathcal{A}}$, in which case $\sim^{\mathfrak{B}} b \in D^{\mathcal{B}}$, while $\pi_0(\pi_i(\sim^{\mathfrak{B}}\sim^{\mathfrak{B}} a)) = 0$, and so, by Lemma 8.3(iii)**d)**, $\pi_0(\pi_i(a(\sqsupset^{\sim}_{\bar\wedge,\veebar})^{\mathfrak{B}}\sim^{\mathfrak{B}}\sim^{\mathfrak{B}} a)) = 0$, in which case $(a(\sqsupset^{\sim}_{\bar\wedge,\veebar})^{\mathfrak{B}}\sim^{\mathfrak{B}}\sim^{\mathfrak{B}} a)) \notin D^{\mathcal{B}}$, and so (8.4) is not true in $\mathcal{B}$ under $[x_0/a]$. Thus, (8.4) is not true in $\mathcal{B} \in \mathrm{Mod}(C)$, and so, by Lemma 8.3(iii)**e)**, $\mathcal{B} \notin \mathrm{Mod}(C^{\mathrm{DMP}})$.

(ii) Assume $\mathcal{B}$ is generated by any (non-empty, unless Σ contains a nullary connective) $S \subseteq K_6^I$ such that $(\{|I|, |S|\} \cap 2) \neq \varnothing$. Consider any $b \in (B \setminus K_5^I)$, in which case there are some $i \in I$ such that $\pi_i(b) \in (K_6 \setminus K_5)$, some $n \in (\omega(\setminus 1))$ (unless Σ contains a nullary connective), some $\phi \in \mathrm{Fm}^n_\Sigma$ and some $\bar{a} \in S^n$ such that $b = \phi^{\mathfrak{B}}(\bar{a})$, and so $\langle \frac{1}{2}, 0\rangle = \pi_i(b) = \phi^{\mathfrak{K}_6}(\pi_i \circ \bar{a})$. Then, as $K_5 \subseteq K_6$ forms a subalgebra of $\mathfrak{A}^2$, and so of $\mathfrak{K}_6$, there must be some $m \in n$ such that $\pi_i(a_m) \notin K_5$, in which case $\pi_i(a_m) = \langle \frac{1}{2}, 0\rangle = \pi_i(b)$, and so $b = a_m \in S$, whenever $|I| \in 2$. Now, assume $|I| \notin 2$, in which case $|S| \in 2$, and so $S = \{a_m\}$. Then, $\varphi \triangleq (\phi[x_l/x_0]_{l\in n}) \in \mathrm{Fm}^1_\Sigma$, in which case $b = \varphi^{\mathfrak{B}}(a_m)$, and so $\langle \frac{1}{2}, 0\rangle = \pi_i(b) = \varphi^{\mathfrak{A}^2}(\pi_i(a_m)) = \varphi^{\mathfrak{A}^2}(\langle \frac{1}{2}, 0\rangle)$. In particular, both $\varphi^{\mathfrak{A}}(\frac{1}{2}) = \frac{1}{2}$ and $\varphi^{\mathfrak{A}}(0) = 0$. And what is more, as 2 forms a subalgebra of $\mathfrak{A}$, $\varphi^{\mathfrak{A}}(1) \in 2$, in which case $\varphi^{\mathfrak{A}}(1) = 1$, for, otherwise, we would have $\varphi^{\mathfrak{A}}(1) = 0$, in which case we would get $\varphi^{\mathfrak{A}^2}(\langle \frac{1}{2}, 1\rangle) = \langle \frac{1}{2}, 0\rangle \notin K_5$, and so $K_5 \ni \langle \frac{1}{2}, 1\rangle$ would not form a subalgebra of $\mathfrak{A}^2$. Thus, $\varphi^{\mathfrak{A}}$ is diagonal, and so $b = a_m \in S$.

(iii) is by (i,ii). □

Claim 8.5. *Suppose $\mathcal{A}$ is both conjunctive and disjunctive, while K_5 forms a subalgebra of $\mathfrak{A}^2$, in which case $\mathcal{A}$ is classically-hereditery, and so K_6 forms a subalgebra of $\mathfrak{A}^2$. Let I be a set and $\mathcal{D}$ a submatrix of $\mathcal{K}_6^I$. Then, $\mathcal{D} \in \mathrm{Mod}(\mathrm{Cn}_{\mathcal{K}_5})$ iff $D \subseteq K_5^I$.*

Proof. The "if" part is immediate. Conversely, assume $\mathcal{D} \in \mathrm{Mod}(\mathrm{Cn}_{\mathcal{K}_5})$. We prove that $D \subseteq K_5^I$, by contradiction. For suppose $D \nsubseteq K_5^I$. Take any $a \in (D \setminus K_5^I) \neq \varnothing$. Then, the submatrix $\mathcal{B}$ of $\mathcal{D}$ generated by a is a finitely-generated model of $\mathrm{Cn}_{\mathcal{K}_5}$, in which case, by Lemmas 2.2, 8.3(iii)**d)**, Corollary 3.10 and Remark 4.3, there are some set J, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{K}_5)^J$, some subdirect product $\mathcal{E}$ of it and some isomorphism e from $\mathcal{B}$ onto $\mathcal{E}$. Moreover, for some $i \in I$, $\pi_i(a) = \langle \frac{1}{2}, 0\rangle \notin D^{\mathcal{K}_6}$, in which case $a \notin D^{\mathcal{B}}$, that is, $e(a) \notin D^{\mathcal{E}}$, and so there is some $j \in J$ such that $\pi_j(e(a)) \in (K_5 \setminus D^{\mathcal{K}_5}) \subseteq 2^2$. Therefore, as 2^2 forms a subalgebra of $\mathfrak{K}_5$, for $\pi_1[K_5] = 2$ forms a subalgebra of $\mathfrak{A}$, $h \triangleq (e \circ \pi_j)$, being a homomorphism from $\mathfrak{B}$ to $\mathfrak{K}_5$ with $(\mathrm{img}\, h) \subseteq 2^2$, for $h(a) \in 2^2$, is a strict homomorphism from $\mathcal{B}' \triangleq \langle \mathfrak{B}, h^{-1}[D^{\mathcal{K}_5}]\rangle$ to the model $(\mathcal{K}_5{\restriction}2^2) = (\mathcal{A}{\restriction}2)^2$ of of the logic of $\mathcal{A}{\restriction}2$, in which case $\mathcal{B}'$ is a finitely-generated model of the logic of $\mathcal{A}{\restriction}2$, and so, by Lemmas 2.2, 8.3(iii)**d)**, Corollary 3.10 and Remark 4.3, $\mathcal{B}' \in \mathbf{I}(\mathbf{P}^{\mathrm{SD}}(\mathbf{S}(\mathcal{A}{\restriction}2)))$. Furthermore, by

Lemma 8.3(iii)**b)**, $g \triangleq ((\pi_i \restriction B) \circ \pi_0) \in \hom(\mathfrak{B}, \mathfrak{A})$ is surjective, for $g(a) = \frac{1}{2}$. Hence, $\mathfrak{A}$ belongs to the variety generated by $\mathfrak{A}\restriction 2$, that is, satisfies any Σ-identity, being true in $\mathfrak{A}\restriction 2$ (in particular, $\sim\sim x_0 \approx x_0$). However, by Lemma 8.3(iii)**a)**, $\sim\sim x_0 \approx x_0$ is not true in $\mathfrak{A}$ under $[x_0/\frac{1}{2}]$. This contradiction completes the argument. □

Let $\mathcal{B}$ be the submatrix of $\mathcal{K}_6^2$ generated by $\langle\langle \frac{1}{2}, 0\rangle, \langle \frac{1}{2}, 1\rangle\rangle \in (\Bbbk_2 \setminus K_5^2) \subseteq K_6^I$. Then, by Claim 8.4(iii)/8.5 with $I = 2$, $\mathcal{B} \in / \notin \mathrm{Mod}(C^{\mathrm{DMP}} / \mathrm{Cn}_{\mathcal{K}_5})$, respectively. In this way, **f)** completes the argument.

j) Then, by **a)**, we have $(\sim^{\mathfrak{A}^2})^{2-i}\langle \frac{1}{2}, 0/1\rangle = \langle i, i/(1-i)\rangle$, for all $i \in 2$, while, with using also the $\veebar$-disjunctivity of $\mathcal{A}\restriction 2$, we get $(\langle \frac{1}{2}, 0\rangle \veebar^{\mathfrak{A}^2} \langle 1, 1\rangle) = \langle \frac{1}{2}, 1\rangle$.

k) (a) is by **d)**, **g)**, Remark 4.3 and the $\bar{\wedge}$-conjunctivity of $\mathcal{A}$.

(b) In case $(\ker \chi_A^2) \in \mathrm{Con}(\mathfrak{A})$, $h \triangleq \chi_A^2$ is a surjective homomorphism from $\mathfrak{A}$ onto its homomorphic image $\mathfrak{B} \triangleq h[\mathfrak{A}]$, in which the quasi-identity $(\sim\sim x_0 \approx x_0) \to (x_0 \approx x_1)$, being true in $\mathfrak{A}$, in view of **a)**, and so in Q, is not true under $[x_i/(1-i)]_{i \in 2}$, and so $\mathsf{Q} \not\ni \mathfrak{B}$ is not a variety, for it is not closed under taking homomorphic images of its members. {Otherwise, $\mathfrak{A}$ is *simple* (i.e., has more than one element but no non-diagonal congruence other than the direct square of its carrier), for the only non-diagonal equivalence relations on A, distinct from both A^2 and $\ker \chi_A^2$, are $(\ker \chi_A^{\{\frac{1}{2},1\}}) \notin \mathrm{Con}(\mathfrak{A})$, because, by **a)**, $\langle \frac{1}{2}, 1\rangle \in (\ker \chi_A^{\{\frac{1}{2},1\}}) \not\ni \langle 1, 0\rangle = \langle \sim^{\mathfrak{A}} \frac{1}{2}, \sim^{\mathfrak{A}} 1\rangle$, and $(\ker \chi_A^{\{\frac{1}{2},0\}}) \notin \mathrm{Con}(\mathfrak{A})$, because, by **a)** and the $\veebar$-disjunctivity of $\mathcal{A}\restriction 2$, $\langle \frac{1}{2}, 0\rangle \in (\ker \chi_A^{\{\frac{1}{2},0\}}) \not\ni \langle \frac{1}{2}, 1\rangle = \langle \frac{1}{2} \veebar^{\mathfrak{A}} 1, 0 \veebar^{\mathfrak{A}} 1\rangle$. In that case, as $\mathfrak{A}\restriction 2$, being two-element, is simple too, by Theorems 1.3, 2.3, 2.6 and the remark 2 after Theorem 2.5 of [10] as well as Corollary 2.3 of [2], the quasi-variety $\mathbb{Q} \supseteq \mathsf{S}$ is the variety generated by S.} (Finally, if $\mathcal{A}$ is classically-valued, then $(\ker \chi_A^2) \in \mathrm{Con}(\mathfrak{A})$.)

(c) Then, according to (i)(A)⇒(C),(iii)**f)**,**i)**(1), the logics of $\mathcal{K}_6$, $\mathcal{K}_5$ and $\mathcal{A}\restriction 2$ are three distinct finitely-valued (and so finitary) consistent proper extensions of C. On the other hand, if $\boldsymbol{\tau}_{\mathsf{S}}^{\mathbb{Q}}$ was injective, then by **g)** and **h)** collectively with Theorem 3.3 of [15] as well as both Theorem 4.4 and Example B.2 of [19], C would have at most two distinct finitary consistent proper extensions. (In this way, Corollary 4.13 and the proof of Corollary 4.12 of [19] as well as [6] {cf. Theorem 2.4 of [10]} complete the argument.) □

Corollary 8.6. *Suppose $\mathcal{A}$ is both $\bar{\wedge}$-conjunctive and $\veebar$-disjunctive, while both $(K|L)_5$ form subalgebras of $\mathfrak{A}^2$. Then, $\mathcal{L}_5 \notin \mathrm{Mod}(C^{\mathrm{DMP}})$.*

Proof. In that case, by Lemma 8.3(iii)**a)**, (8.4) is not true in $\mathcal{L}_5$ under $[x_0/\langle \frac{1}{2}, 0\rangle]$, for $\mathcal{A}$ is $\sqsupset$-implicative, where $\sqsupset = \sqsupset^{\sim}_{\bar{\wedge},\veebar}$, in view of Lemma 8.3(iii)**d)**. In this way, Lemma 8.3(iii)**e)** completes the argument. □

In view of Remark 4.2, conjunctive/disjunctive/implicative three-valued $\sim$-paraconsistent Σ-logics with subclassical negation $\sim$ (in particular, P^1; cf. Subsubsection 4.1.3) are covered by both Lemma 8.3(iii) and Corollary 8.6, in which case, in particular, Lemma 8.3(iii)**k)**(a,b) subsumes Lemma 4.1 and Theorem 4.6 of [12].

And what is more, Lemma 8.3(iii) covers, in particular, $NIHZ$ (cf. Paragraph 4.1.2.1), for, in that case, 2 forms a subalgebra of $\mathfrak{A}$ (in particular, $2^2 \subseteq K_5$ forms a subalgebra of $\mathfrak{A}^2$), while, by Lemma 8.3(iii)**a)**, $\sim^{\mathfrak{A}}\langle\frac{1}{2},1\rangle = \langle 1,0\rangle \in K_5$, whereas $\mathfrak{K}_6{\restriction}\Sigma_+$ is a $(\wedge,\vee)$-lattice with unit $\langle\frac{1}{2},1\rangle \in K_5$, in view of the $\vee$-disjunctivity of $\mathcal{A}{\restriction}2$ and the fact that $\flat^{\mathfrak{A}}_{\veebar} = \frac{1}{2}$, with $\mathbb{Q}$ not being a variety, in view of Lemma 8.3(iii)**k)**(b), for $\theta \triangleq (\ker\chi^2_A)$ is a congruence of the chain lattice $\mathfrak{A}{\restriction}\Sigma_+$, because 2 is an ideal of it, while $(\theta \setminus \Delta_A) = L_2$, whereas $\sim^{\mathfrak{A}^2}[L_2] \subseteq L_2 \subseteq \theta$, but neither of the rest of instances discussed in Subsection 4.1 are covered by Lemma 8.3(iii), in view of Lemma 8.3(iii)**a)**. On the other hand, there are instances, covered by Lemma 8.3(iii), with $\mathbb{Q}$ being a variety, in view of:

Example 8.7. Let $\Sigma \triangleq \Sigma^{\supset}_{\sim,+}$ and C the three-valued Σ-expansion of $NIHZ$ (cf. Corollary 4.6 and Subsubsection 4.1.2.1) given by:

$$(a \supset^{\mathfrak{A}} b) \triangleq \begin{cases} a & \text{if } a = b, \\ \frac{1}{2} & \text{if } \{a,b\} = \{0,\frac{1}{2}\}, \\ 1 & \text{otherwise,} \end{cases}$$

for all $a, b \in A$. Then, 2 forms a subalgebra of $\mathfrak{A}$, in which case 2^2 forms a subalgebra of $\mathfrak{A}^2$, and so does K_5, because, for all $a \in K_5$, $\pi_1(\langle\frac{1}{2},1\rangle \supset^{\mathfrak{A}^2} a) = 1 = \pi_1(a \supset^{\mathfrak{A}^2} \langle\frac{1}{2},1\rangle)$ (in particular, $\{\langle\frac{1}{2},1\rangle \supset^{\mathfrak{A}^2} a, a \supset^{\mathfrak{A}^2} \langle\frac{1}{2},1\rangle\} \subseteq (A \times \{1\}) \subseteq K_5$), while K_5 forms a subalgebra of $(\mathfrak{A}{\restriction}\Sigma_{\sim,+})^2$, as it has been argued above. Moreover, $\theta \triangleq (\ker\chi^2_A) \notin \mathrm{Con}(\mathfrak{A})$, for $\langle 0,1\rangle \in \theta \not\ni \langle\frac{1}{2},1\rangle = \langle\frac{1}{2} \supset^{\mathfrak{A}} 0, \frac{1}{2} \supset^{\mathfrak{A}} 1\rangle$. Hence, by Lemma 8.3(iii)**k)**(b), the quasivariety generated by $\mathfrak{A}$ is a variety. □

And what is more, these two instances collectively with the following generic observation show that the stipulation $(\{|I|,|S|\} \cap 2) \neq \varnothing$ cannot be omitted in the formulation of Claim 8.4(ii/iii):

Remark 8.8. Suppose $\mathcal{A}$ is both $\bar{\wedge}$-conjunctive and $\veebar$-disjunctive, while $\mathfrak{A}$ is a $(\bar{\wedge},\veebar)$-lattice, whereas K_5 forms a subalgebra of $\mathfrak{A}^2$, in which case $\mathcal{A}$ is classically-hereditary, and so K_6 forms a subalgebra of $\mathfrak{A}^2$. Let I be a set, $\bar{\imath} \in (I^2 \setminus \Delta_I)$, $a_j \triangleq (\{\langle i_j, \langle 1,0\rangle\rangle\} \cup ((I \setminus \{i_j\}) \times \{\langle\frac{1}{2},0\rangle\})) \in \Bbbk_I$, where $j \in 2$, and $\mathcal{B}$ the submatrix of $\mathcal{K}^I_6$ generated by $S \triangleq (\mathrm{img}\,\bar{a})$. Then, by Lemma 8.3(iii)**a)** and the idempotencity of $\veebar^{\mathfrak{A}}$, $B \ni (\veebar^{\mathfrak{B}}\bar{a}) = (I \times \{\langle\frac{1}{2},0\rangle\}) \notin \Bbbk_I \supseteq K^I_5$, in which case $(B \setminus K^I_5) \nsubseteq \Bbbk_I \supseteq S$, and so $(B \setminus K^I_5) \nsubseteq S$, while, by Claim 8.4(i), $\mathcal{B} \notin \mathrm{Mod}(C^{\mathrm{DMP}})$. □

Finally, Lemma 8.3(iii)**j)** yields a one more insight into the fact that the scopes of the next two subsections are disjoint.

8.1. **3VLPSN with lattice conjunction and disjunction.** Here, it is supposed that $\mathcal{A}$ (viz., C) is $\bar{\wedge}$-conjunctive, while $\mathfrak{A}$ is a $(\bar{\wedge},\veebar)$-lattice, in which case it is that with zero 0, for $\mathcal{A}$ is false-singular with non-distinguished value 0, while both $\bar{\wedge}$ and $\veebar$ are both symmetric and idempotent formulas for $\mathcal{A}$, whereas the poset $\langle A, \leq^{\mathfrak{A}}_{\bar{\wedge}}\rangle$ is a chain, for $|A| = 3$ is finite, and so $\mathcal{A}$ is $\veebar$-disjunctive (in particular, C is so, while $\mathfrak{A}$ is a distributive $(\bar{\wedge},\veebar)$-lattice with unit in $D^{\mathcal{A}}$, for this is non-empty). Then, by the $\veebar$-disjunctivity of $\mathcal{A}$ and the fact that $(A \setminus D^{\mathcal{A}}) = \{\flat^{\mathfrak{A}}_{\bar{\wedge}}\}$, we have $(\sim(x_0 \vee x_1) \vee x_1) \in C(\{x_0 \vee x_1, \sim x_0 \vee x_1\})$, and so get:

$$C^{\mathrm{R}} = C^{\mathrm{MP}}. \tag{8.5}$$

Lemma 8.9. *Let I be a finite set, $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and $\mathcal{D}$ a consistent non-$\sim$-paraconsistent subdirect product of it. Then, $\mathcal{A}$ is classically-hereditary, while* $\hom(\mathcal{D},\mathcal{A}{\restriction}2) \neq \varnothing$*.*

Proof. First of all, we prove, by contradiction, that there is some $i \in I$ such that $\frac{1}{2} \notin C_i$. For suppose $\frac{1}{2} \in C_i$, for each $i \in I$. Then, as the poset $\langle A, \leq^{\mathfrak{A}}_{\overline{\wedge}} \rangle$ is a chain, we have both $\frac{1}{2}(\leq / \geq)^{\mathfrak{A}}_{\overline{\wedge}} {\sim}^{\mathfrak{A}} \frac{1}{2}$ and $\frac{1}{2}(\leq | \geq)^{\mathfrak{A}}_{\overline{\wedge}} 1$. By induction on the cardinality of any $J \subseteq I$, we prove that there is some $a \in (D \cap \{\frac{1}{2}, 0/1\}^I)$ including $J \times \{\frac{1}{2}\}$. The case, when $J = \varnothing$, is by Lemma 3.1 /"and the fact that ${\sim}^{\mathfrak{A}} 0 = 1$". Otherwise, take any $j \in J \subseteq I$, in which case $K \triangleq (J \setminus \{j\}) \subseteq I$, while $|K| < |J|$, and so, by the induction hypothesis, there is some $b \in (D \cap \{\frac{1}{2}, 0/1\}^I)$ including $K \times \{\frac{1}{2}\}$. Moreover, as $j \in C_j = \pi_j[D]$, there is some $c \in D$ such that $\pi_j(c) = \frac{1}{2}$, in which case $D \ni d \triangleq (c(\overline{\wedge}/\underline{\vee})^{\mathfrak{D}} {\sim}^{\mathfrak{D}} c)$, and so, for all $i \in I$, $\pi_i(d) = \frac{1}{2}$, if $\pi_i(c) = \frac{1}{2}$ (in particular, $\pi_j(d) = \frac{1}{2}$), and $\pi_i(d) = (0/1)$, otherwise. Then, $D \in a \triangleq (b(\underline{\vee}/(\overline{\wedge}|\underline{\vee}))^{\mathfrak{A}} d)$, in which case, since $0 \leq^{\mathfrak{A}}_{\overline{\wedge}} \frac{1}{2}(\leq | \geq)^{\mathfrak{A}}_{\overline{\wedge}} 1$, while $b \in \{\frac{1}{2}, 0/1\}^I$, for all $i \in I$, $\pi_i(a) = \frac{1}{2}$, if either $\pi_i(b) = \frac{1}{2}$ or $\pi_i(d) = \frac{1}{2}$ (in particular, $\pi_k(d) = \frac{1}{2}$, for all $k \in (K \cup \{j\}) = J$), and $\pi_i(a) = (0/1)$, otherwise, and so $a \in \{\frac{1}{2}, 0/1\}^I$ includes $J \times \{\frac{1}{2}\}$, as required. In particular, when $J = I$, there is some $a \in D$ including $I \times \{\frac{1}{2}\}$, and so equal to this, for $I = (\operatorname{dom} a)$, in which case $\{a, {\sim}^{\mathfrak{D}} a\} \subseteq D^{\mathcal{D}}$, and so $\mathcal{D}$, being consistent, is $\sim$-paraconsistent. This contradiction shows that there is some $i \in I$ such that $\frac{1}{2} \notin C_i \subseteq A$, in which case, since $0 \in C_i$, for C_i is a consistent submatrix of $\mathcal{A}$, being false-singular with non-distinguished value 0, and so $1 = {\sim}^{\mathfrak{A}} 0 \in C_i$, $C_i = 2$ forms a subalgebra of $\mathfrak{A}$, while $\mathcal{C}_i = (\mathcal{A} \restriction C_i)$, whereas $(\pi_i \restriction D) \in \hom(\mathcal{D}, \mathcal{C}_i)$, as required. □

Theorem 8.10. *Providing C is $\sim$-subclassical (i.e,, $\mathcal{A}$ is classically-hereditary; cf. Corollary 6.5), C^{NP} is defined by $\mathcal{K}_6$, in which case $C^{\mathrm{NP}} = C_\omega$, and so $C_\omega = C_n$, for all $n \in (\omega \setminus 1)$.*

Proof. Then, By Theorem 2.3, C^{NP} is finitely-equivalent to the logic C' of the class S of all consistent members of $\mathbf{P}^{\mathrm{SD}}_{\omega}(\mathbf{S}_*(\mathcal{A})) \cap \mathrm{Mod}(C^{\mathrm{NP}})$. Consider any $\mathcal{B} \in \mathsf{S} \subseteq \mathrm{Mod}\,(2.9)$, in which case there are some finite set I and some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ such that $\mathcal{B}$ is a subdirect product of it, and so, by Lemma 8.9, there is some $g \in \hom(\mathcal{B}, \mathcal{A} \restriction 2)$. Consider any $a \in (B \setminus D^{\mathcal{B}})$. Then, there is some $i \in I$ such that $\pi_i(a) \notin D^{\mathcal{B}}$, in which case $h : B \to (A \times 2), b \mapsto \langle \pi_i(b), g(b) \rangle$ belongs to $J \triangleq \hom(\mathcal{B}, \mathcal{K}_6)$, while $h(a) \notin D^{\mathcal{K}_6}$, and so $e : B \to (A \times 2)^J, b \mapsto \langle f(b) \rangle_{f \in J}$ is a strict homomorphism from $\mathcal{B}$ to $\mathcal{K}_6^J$. Hence, $\mathcal{B}$ is a model of the logic of $\mathcal{K}_6$, in which case this is a sublogic of C', and so, being six-valued (in particular, finitary), is a sublogic of C^{NP}, as required, in view of Remark 8.2, Lemma 8.3(iii)(B)⇒(C), the $\underline{\vee}$-disjunctivity of $\mathcal{A}$ and the fact that $C^{\mathrm{NP}} = C_1 \subseteq C_\omega$. □

Remark 8.11. $\nabla^{\sim}_{\underline{\vee}\sim}$ defines truth in $\mathcal{A}$, whenever ${\sim}^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$. In particular, if $\mathcal{A}$ is $\sqsupset$-implicative, then, by Remark 4.3 and the $\overline{\wedge}$-conjunctivity of $\mathcal{A}$, $\varepsilon^{\sim}_{\sqsupset,\overline{\wedge}}$ is an axiomatic canonical equality determinant for $\mathcal{A}$, in which case C is equivalent to the quasivariety generated by $\mathfrak{A}$ with respect to $\nabla^{\sim}_{\underline{\vee}\sim}$ and $\varepsilon^{\sim}_{\sqsupset,\overline{\wedge}}$ in the sense of [14], while $\nabla^{\sim}_{\underline{\vee}\sim}[x_0/(\varepsilon^{\sim}_{\sqsupset,\overline{\wedge}} \sqsupset (\varepsilon^{\sim}_{\sqsupset,\overline{\wedge}}[x_i/x_{2+i}]_{i \in 2}))]$ is an implicative system for $\mathfrak{A}$, and so the quasivariety involved is implicative. □

This covers arbitrary three-valued extensions of LP as well as, in the "implicative" case, those of both $\mathbb{S}_3$ and HZ.

Despite of Remark 8.11 and Lemma 8.3(iii)**g)**, truth need not, generally speaking, be equationally definable in $\mathcal{A}$, in view of:

Example 8.12. Let $C \triangleq PG3^{[*]}$ (cf. Paragraph 4.1.4.2 [resp. 4.1.4.1]). Then,

$$h \triangleq \chi^{\{1\}}_A \in \hom(\mathfrak{A}, \mathfrak{A}). \tag{8.6}$$

Therefore, if any $\nabla \subseteq \mathrm{Eq}^1_\Sigma$ defined truth in $\mathcal{A}$, then, as $\frac{1}{2} \in D^{\mathcal{A}}$, we would have $\mathfrak{A} \models (\bigwedge \nabla)[x_0/\frac{1}{2}]$, in which case, by (8.6), since $h(\frac{1}{2}) = 0$, we would get $\mathfrak{A} \models (\bigwedge \nabla)[x_0/0]$, and so would eventually get $0 \in D^{\mathcal{A}}$. Hence, truth is not equationally definable in $\mathcal{A}$. This ensues from the following observation as well, equally showing that the condition of the equational definability of truth in defining matrices is essential for Theorem 3.3 of the study [15] (to which the reader is referred for the conception of *prevariety*) to hold. Namely, by Corollary 6.5 and the double-optional version of Theorem 6.8(ii)⇒(iii), C^{NP} and C^{PC} are two distinct consistent proper extensions of C. On the other hand, the prevariety generated by $\mathfrak{A}{\restriction}2$, being the subprevariety of the prevariety P generated by $\mathfrak{A}$ (i.e., constituted by isomorphic copies of subdirect products of tuples consisting of non-one-elements subalgebras of $\mathfrak{A}$), relatively axiomatized by the Σ-identity:

$$(x_0 \wedge {\sim}x_0) \approx \bot, \tag{8.7}$$

for $\mathfrak{A}{\restriction}2$ is the only subalgebra of $\mathfrak{A}$ satisfying (8.7), is the only *non-trivial* (viz., containing a non-one-element member) subprevariety of P, for $\mathfrak{A}{\restriction}2$ is clearly embeddable into any non-one-element member of P (more precisely, given any set I and any non-one-element subalgebra $\mathfrak{B}$ of $\mathfrak{A}^I$, we have $B \ni \bot^{\mathfrak{B}} = (I \times \{0\}) \neq (I \times \{1\}) = \top^{\mathfrak{B}} \in B$, in which case $I \neq \varnothing$, and so $\{\langle k, I \times \{k\}\rangle \mid k \in 2\}$ is an embedding of $\mathfrak{A}{\restriction}2$ into $\mathfrak{B}$ [in particular, $\mathfrak{A}{\restriction}2$ is embeddable into any isomorphic copy of $\mathfrak{B}$]), while, with using (8.6), $\mathfrak{A}$ is easy to see to be embeddable into any member of P not satisfying (8.7) (more precisely, given any set I and any subalgebra $\mathfrak{B}$ of $\mathfrak{A}^I$ not satisfying (8.7), there is some $a \in B$ such that $(B \cap \{0, \frac{1}{2}\}^I) \ni b \triangleq (a \wedge^{\mathfrak{B}} {\sim}^{\mathfrak{B}} a) \neq \bot^{\mathfrak{B}} = (I \times \{0\}) \in B \ni \top^{\mathfrak{B}} = (I \times \{1\})$, in which case $J \triangleq \{i \in I \mid \pi_i(b) = \frac{1}{2}\} \neq \varnothing$, and so, by (8.6), $\{\{\langle c, (J \times \{c\}) \cup ((I \setminus J) \times \{h(c)\})\rangle \mid c \in A\}$ is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$ [in particular, $\mathfrak{A}$ is embeddable into any isomorphic copy of $\mathfrak{B}$]). In this way, by Theorem 3.3 of [15], if either truth was definable in $\mathcal{A}$ or the condition of the equational definability of truth in defining matrices was redundant in the formulation of the mentioned theorem, then C would have at most one consistent proper extension. □

Lemma 8.13. *Let C' be an extension of C. Suppose C is ${\sim}$-subclassical (i.e., 2 forms a subalgebra of $\mathfrak{A}$, C^{PC} being defined by $\mathcal{A}{\restriction}2$; cf. Corollary 6.5), while* (3.2) *is not satisfied in C'. Then, $\mathcal{K}_5' \in \mathrm{Mod}(C')$. In particular, $C^{\mathrm{DMP}} = C^{\mathrm{PC}}$, unless K_5 forms a subalgebra of $\mathfrak{A}^2$.*

Proof. Then, by Theorem 8.10, C^{NP} is defined by $\mathcal{K}_6$. On the other hand, as C' does not satisfy the finitary (3.2), by Theorem 2.3, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathrm{Mod}(C')$ of it not being a model of (3.2), in which case there are some $a \in D^{\mathcal{D}} \subseteq \{\frac{1}{2}, 1\}^I$ and some $b \in (D \setminus D^{\mathcal{D}})$ such that $({\sim}^{\mathfrak{D}} a \veebar^{\mathfrak{D}} b) \in D^{\mathcal{D}}$, and so $J \triangleq \{i \in I \mid \pi_i(a) = \frac{1}{2}\} \supseteq K \triangleq \{i \in I \mid \pi_i(b) = 0\} \neq \varnothing$. Put $L \triangleq \{i \in I \mid \pi_i(b) = 1\}$. Then, given any $\bar{a} \in A^5$, set $(a_0 : a_1 : a_2 : a_3 : a_4) \triangleq ((((I \setminus (L \cup K)) \cap J) \times \{a_0\}) \cup ((I \setminus (L \cup J)) \times \{a_1\}) \cup ((L \setminus J) \times \{a_2\}) \cup ((L \cap J) \times \{a_3\}) \cup (K \times \{a_4\})) \in A^I$. In this way:

$$D \ni a = (\tfrac{1}{2} : 1 : 1 : \tfrac{1}{2} : \tfrac{1}{2}), \tag{8.8}$$

$$D \ni b = (\tfrac{1}{2} : \tfrac{1}{2} : 1 : 1 : 0). \tag{8.9}$$

Moreover, by Lemma 3.1, we also have:

$$D \ni f \triangleq (0 : 0 : 0 : 0 : 0), \tag{8.10}$$

$$D \ni t \triangleq {\sim}^{\mathfrak{D}} f = (1 : 1 : 1 : 1 : 1). \tag{8.11}$$

Consider the following exhaustive (as ${\sim}^{\mathfrak{A}} \frac{1}{2} \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$) cases:

- ${\sim}^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$.
  Then, in case $\frac{1}{2} \leq^{\mathfrak{A}}_{\wedge} 1$, by (8.8) and (8.9), we have:

$$D \ni e \triangleq (a \mathbin{\overline{\wedge}^{\mathfrak{D}}} b) = (\tfrac{1}{2} : \tfrac{1}{2} : 1 : \tfrac{1}{2} : 0), \tag{8.12}$$

$$D \ni {\sim}^{\mathfrak{D}} e = (\tfrac{1}{2} : \tfrac{1}{2} : 0 : \tfrac{1}{2} : 1), \tag{8.13}$$

$$D \ni c \triangleq (e \veebar^{\mathfrak{D}} {\sim}^{\mathfrak{D}} b) = (\tfrac{1}{2} : \tfrac{1}{2} : 1 : \tfrac{1}{2} : 1), \tag{8.14}$$

$$D \ni {\sim}^{\mathfrak{D}} c = (\tfrac{1}{2} : \tfrac{1}{2} : 0 : \tfrac{1}{2} : 0). \tag{8.15}$$

Likewise, in case $\frac{1}{2}(\leq_{\bar{\wedge}} / \geq)^{\mathfrak{A}} 1$, by (8.8) and (8.12)/(8.9), we have:

$$D \ni d \triangleq ((e/b) \veebar^{\mathfrak{D}} {\sim}^{\mathfrak{D}} a) = (\tfrac{1}{2} : \tfrac{1}{2} : 1 : \tfrac{1}{2} : \tfrac{1}{2}), \tag{8.16}$$

$$D \ni {\sim}^{\mathfrak{D}} d = (\tfrac{1}{2} : \tfrac{1}{2} : 0 : \tfrac{1}{2} : \tfrac{1}{2}). \tag{8.17}$$

Consider the following complementary subcases:

- $L \subseteq J$.
  Then, since $I \supseteq K \neq \varnothing = (L \setminus J)$, by (8.10), (8.11) and (8.16), $\langle g, I \times \{g\}\rangle \mid g \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{K}'_5 \in \mathrm{Mod}(C)$.
- $L \nsubseteq J$.
  Then, consider the following complementary subsubcases:
  * there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2}, 0) = 0$ and $\varphi^{\mathfrak{A}}(0,0) = 1$, in which case, by (8.10) and (8.17), we have:

$$D \ni \varphi^{\mathfrak{D}}({\sim}^{\mathfrak{D}} d, f) = (0 : 0 : 1 : 0 : 0), \tag{8.18}$$

$$D \ni {\sim}^{\mathfrak{D}} \varphi^{\mathfrak{D}}({\sim}^{\mathfrak{D}} d, f) = (1 : 1 : 0 : 1 : 1). \tag{8.19}$$

    Then, since $(L \setminus J) \neq \varnothing \neq K$, taking (8.10), (8.11), (8.16), (8.17), (8.18) and (8.19) into account, we see that $\{\langle\langle g, h\rangle, (g : g : h : g : g)\rangle \mid \langle g, h\rangle \in K_6\}$ is an embedding of $\mathcal{K}_6$ into $\mathcal{D}$, and so $\mathcal{K}_6$ is a model of C', for $\mathcal{D}$ is so, and so is its submatrix $\mathcal{K}'_5$, for $K_6 \supseteq K_5$ forms a subalgebra of $\mathfrak{A}^2$, because 2 forms a subalgebra of $\mathfrak{A}$.
  * there is no $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2}, 0) = 0$ and $\varphi^{\mathfrak{A}}(0,0) = 1$, Then, $\frac{1}{2} \leq^{\mathfrak{A}}_{\bar{\wedge}} 1$, for, otherwise, we would have $1 \leq^{\mathfrak{A}}_{\bar{\wedge}} \frac{1}{2}$, in which case we would get $\varphi^{\mathfrak{A}}(\frac{1}{2}, 0) = 0$ and $\varphi^{\mathfrak{A}}(0,0) = 1$, where $\varphi \triangleq {\sim}(x_0 \bar{\wedge} {\sim} x_1) \in \mathrm{Fm}^2_\Sigma$. Consider the following complementary subsubsubcases:
    · $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) = \varnothing$.
      Then, taking (8.12), (8.13), (8.14), (8.15), (8.16) and (8.17) into account, as $K \neq \varnothing \neq (L \setminus J)$, we conclude that $\{\langle\langle g, h\rangle, (\frac{1}{2} : \frac{1}{2} : h : \frac{1}{2} : g)\rangle \mid \langle g, h\rangle \in L_6\}$ is an embedding of $\mathcal{K}_6$ into $\mathcal{D}$, and so $\mathcal{K}_6$ is a model of C', for $\mathcal{D}$ is so, and so is its submatrix $\mathcal{K}'_5$, for $K_6 \supseteq K_5$ forms a subalgebra of $\mathfrak{A}^2$, because 2 forms a subalgebra of $\mathfrak{A}$.
    · $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) \neq \varnothing$.
      Let $\mathfrak{G}$ be the subalgebra of $\mathfrak{K}_6 \times \mathfrak{A}$ generated by $((K_6 \times \{\frac{1}{2}\}) \cup \{\langle\langle i, i\rangle, i\rangle \mid i \in 2\})$. Then, as $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) \neq \varnothing \notin \{K, L \setminus J\}$, by (8.10), (8.11), (8.12), (8.13), (8.14), (8.15), (8.16) and (8.17), we see that $\{\langle\langle\langle g, h\rangle, j\rangle, (j : j : h : j : g)\rangle \mid \langle\langle g, h\rangle, j\rangle \in G\}$ is an embedding of $\mathcal{G} \triangleq ((\mathcal{K}_6 \times \mathcal{A}){\restriction}G)$ into $\mathcal{D}$, in which case $\mathcal{G}$ is a model of C', for $\mathcal{D}$ is so. Let us prove, by contradiction, that $((D^{\mathcal{K}_6} \times \{0\}) \cap G) = \varnothing$. For suppose $((D^{\mathcal{K}_6} \times \{0\}) \cap G) \neq \varnothing$. Then, there is some $\psi \in \mathrm{Fm}^8_\Sigma$ such that $\psi^{\mathfrak{A}}(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0) = 0$ and $\psi^{\mathfrak{A}}(1,1,1,1,0,0,0,0) = 1$, for $\pi_1[D^{\mathcal{K}_6}] = \{1\}$. Let $\varphi \triangleq \psi({\sim}x_1, {\sim}x_0, {\sim}x_0, {\sim}x_0, x_0, x_0, x_0, x_1) \in \mathrm{Fm}^2_\Sigma$. Then, $\varphi^{\mathfrak{A}}(\frac{1}{2}, 0) = 0$ and $\varphi^{\mathfrak{A}}(0,0) = 1$. This

contradiction shows that $((D^{\mathcal{K}_6} \times \{0\}) \cap G) = \varnothing$, in which case $(\pi_0 \restriction G) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{K}_6)$, and so $\mathcal{K}_6$ is a model of C', for $\mathcal{G}$ is so, and so is its submatrix $\mathcal{K}'_5$, for $K_6 \supseteq K_5$ forms a subalgebra of $\mathfrak{A}^2$, because 2 forms a subalgebra of $\mathfrak{A}$.

- $\sim^{\mathfrak{A}} \frac{1}{2} = 1$,
  Consider the following exhaustive (as $\langle A, \leq^{\mathfrak{A}}_{\bar\wedge}\rangle$ is a chain poset) subcases:
  - $\frac{1}{2} \leq^{\mathfrak{A}}_{\bar\wedge} 1$.
    Then, by (8.8) and (8.9), we get:
    $$D \ni c' \triangleq (a \veebar^{\mathfrak{D}} b) = (\tfrac{1}{2} : 1 : 1 : 1 : \tfrac{1}{2}), \tag{8.20}$$
    $$D \ni d' \triangleq \sim^{\mathfrak{D}} c' = (1 : 0 : 0 : 0 : 1), \tag{8.21}$$
    $$D \ni e' \triangleq \sim^{\mathfrak{D}} d' = (0 : 1 : 1 : 1 : 0), \tag{8.22}$$
    $$D \ni f' \triangleq (c' \mathbin{\bar\wedge}^{\mathfrak{D}} d') = (\tfrac{1}{2} : 0 : 0 : 0 : \tfrac{1}{2}). \tag{8.23}$$
    Consider the following complementary subsubcases:
    * $((I \setminus (L \cup J)) \cup (L \setminus J) \cup (L \cap J)) = \varnothing$.
      Then, since $I \supseteq K \neq \varnothing$, by (8.10), (8.11) and (8.20), we see that $\{\langle g, I \times \{g\}\rangle \mid g \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{K}'_5 \in \mathrm{Mod}(C)$.
    * $((I \setminus (L \cup J)) \cup (L \setminus J) \cup (L \cap J)) \neq \varnothing$.
      Then, as $K \neq \varnothing$, by (8.10), (8.11), (8.20), (8.21), (8.22) and (8.23), we conclude that $\{\langle\langle g, h\rangle, (g : h : h : h : g)\rangle \mid \langle g, h\rangle \in K_6\}$ is an embedding of $\mathcal{K}_6$ into $\mathcal{D}$, in which case $\mathcal{K}_6$ is a model of C', for $\mathcal{D}$ is so, and so is its submatrix $\mathcal{K}'_5$, for $K_6 \supseteq K_5$ forms a subalgebra of $\mathfrak{A}^2$, because 2 forms a subalgebra of $\mathfrak{A}$.
  - $1 \leq^{\mathfrak{A}}_{\bar\wedge} \frac{1}{2}$.
    Then, by (8.8) and (8.9), we get:
    $$D \ni c'' \triangleq (a \veebar^{\mathfrak{D}} b) = (\tfrac{1}{2} : \tfrac{1}{2} : 1 : \tfrac{1}{2} : \tfrac{1}{2}), \tag{8.24}$$
    $$D \ni d'' \triangleq \sim^{\mathfrak{D}} c'' = (1 : 1 : 0 : 1 : 1), \tag{8.25}$$
    $$D \ni e'' \triangleq \sim^{\mathfrak{D}} d'' = (0 : 0 : 1 : 0 : 0). \tag{8.26}$$
    Consider the following complementary subsubcases:
    * $L \subseteq J$.
      Then, as $I \supseteq K \neq \varnothing = (L \setminus J)$, taking (8.10), (8.11) and (8.24) into account, we see that $\{\langle g, I \times \{g\}\rangle \mid g \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{K}'_5 \in \mathrm{Mod}(C)$.
    * $L \nsubseteq J$.
      Then, as $K \neq \varnothing \neq (L \setminus J)$, taking (8.10), (8.11), (8.24), (8.25) and (8.26) into account, we see that $\{\langle\langle g, h\rangle, (g : g : h : g : g)\rangle \mid \langle g, h\rangle \in K'_5\}$ is an embedding of $\mathcal{K}'_5$ into $\mathcal{D}$, in which case $\mathcal{K}'_5$ is a model of C', for $\mathcal{D}$ is so.

In this way, (8.5), Corollary 6.7, Theorem 8.10 and Lemma 8.3(i)(B)⇒(A) complete the argument, for $\mathcal{K}'_5 = \mathcal{K}_6$, unless K_5 forms a subalgebra of $\mathfrak{A}^2$, because $(K_6 \setminus K_5) = \{\langle \frac{1}{2}, 0\rangle\}$ is a singleton, while $K_6 \supseteq K_5$ forms a subalgebra of $\mathfrak{A}^2$, since 2 forms a subalgebra of $\mathfrak{A}$. □

Corollary 8.14. *Let C' be an extension of C. Suppose* (8.1) *is not satisfied in C'. Then, $C' \subseteq C^{\mathrm{NP}}$.*

Proof. The case, when C^{NP} is inconsistent, is evident. Now, assume C^{NP} is consistent. Then, by Theorem 6.8, C is $\sim$-subclassical (i.e., 2 forms a subalgebra of

$\mathfrak{A}$, C^{PC} being defined by $\mathcal{A}{\restriction}2$; cf. Corollary 6.5), in which case, by Theorem 8.10, C^{NP} is defined by $\mathcal{K}_6$. Consider the following complementary cases:

- K_5 forms a subalgebra of $\mathfrak{A}^2$.
  Then, as C' does not satisfy the finitary (8.1), by Theorem 2.3, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathrm{Mod}(C')$ of it not being a model of (8.1), in which case there are some $a \in D$ and some $b \in (D \setminus D^{\mathcal{D}})$ such that $(a \veebar^{\mathfrak{D}} b) \in D^{\mathcal{D}} \ni {\sim}^{\mathfrak{D}} a$, in which case $a \in \{\frac{1}{2}, 0\}^I$, and so $J \triangleq \{i \in I \mid \pi_i(a) = \frac{1}{2}\} \supseteq \{i \in I \mid \pi_i(b) = 0\} \neq \varnothing$. Then, given any $\bar{a} \in A^2$, set $(a_0 : a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in A^I$. In this way, $D \ni a = (\frac{1}{2} : 0)$. Consider the following complementary subcases:
  - $J = I$,
    Then, $D \ni a = (I \times \{\frac{1}{2}\})$, in which case, as $I = J \neq \varnothing$, by Lemma 8.3(iii)**b)**, $\{\langle x, I \times \{x\}\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, and so $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so. In this way, $C' \subseteq C \subseteq C^{\mathrm{NP}}$.
  - $J \neq I$,
    Then, as $J \neq \varnothing \neq (I \setminus J)$, by Lemma 8.3(iii)**j)**, $\{\langle\langle x, y\rangle, (x : y)\rangle \mid \langle x, y\rangle \in K_6\}$ is an embedding of $\mathcal{K}_6$ into $\mathcal{D}$, in which case $\mathcal{K}_6$ is a model of C', for $\mathcal{D}$ is so, and so $C' \subseteq C^{\mathrm{NP}}$.
- K_5 does not form a subalgebra of $\mathfrak{A}^2$.
  Then, $\mathcal{K}_5' = \mathcal{K}_6$, for $(K_6 \setminus K_5) = \{\langle \frac{1}{2}, 0\rangle\}$ is a singleton, while $K_6 \supseteq K_5$ forms a subalgebra of $\mathfrak{A}^2$, because 2 forms a subalgebra of $\mathfrak{A}$. And what is more, by (8.5), we have $C^{\mathrm{DMP}} \subseteq C^{\mathrm{MP}}$, in which case (3.2) is not satisfied in C', and so, by Lemma 8.13, we get $C' \subseteq C^{\mathrm{NP}}$. $\Box$

Finally, by Lemmas 3.2, 3.3, 8.3, 8.13, Remark 8.1, Corollaries 5.4, 6.5, 6.7, 6.9, 6.13, 6.19, 8.14, Theorems 6.8, 8.10, 7.8 and (8.5), we eventually get:

Theorem 8.15. *Suppose C is [not] non-${\sim}$-subclassical — i.e., 2 is [not] non-$\mathfrak{A}$-closed — and (not) non-implicative [i.e., (n)either K_3 (n)or K_4 forms a subalgebra of $\mathfrak{A}^2$, while K_5 is ({not}) non-$\mathfrak{A}^2$-closed]. Then, the following hold:*

(i) *[({some of})] extensions of C form the $(2[+2(\{+2\})])$-element chain $C \subsetneq C^{\mathrm{NP}} = [\mathrm{Cn}^{\omega}_{\mathcal{K}_6} \subsetneq] C^{\mathrm{DMP}} = [(\{\subsetneq \mathrm{Cn}^{\omega}_{\mathcal{K}_5} \subsetneq\})](C^{\mathrm{INP}} =) C^{\mathrm{MP|R}} = [C^{\mathrm{PC}} = \mathrm{Cn}_{\mathcal{A}\restriction 2} \subsetneq] C^{\mathrm{IC}}$ [({others being extensions/sublogics of $C^{\mathrm{DMP}}/\mathrm{Cn}_{\mathcal{K}_5}$, respectively})]. In particular, C has no [more than one] extension, not being an extension of C^{NP};*

(ii) *$C[\cup(C^{\mathrm{PC}}(\cap(C^{\mathrm{NP}}\{\cup \mathrm{Cn}^{\omega}_{\mathcal{K}_5}\})))]$ is the structural completion of C. In particular, (8.2) is admissible in C iff [resp., "anyway, while" (resp., iff)] it is derivable in C iff ${\sim}^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$.*

In particular, C has exactly four extensions, forming a chain, C^{PC} being its structural completion, unless it is implicative.

In view of Theorem 4.1, the item (ii) of the above theorem exhausts the issue of finite matrix semantics of the structural completions of ${\sim}$-paraconsistent three-valued Σ-logics with subclassical negation ${\sim}$ as well as lattice conjunction and disjunction. And what is more, its item (i) subsumes the particular results, thus providing a generic insight into these, obtained *ad hoc* for LP in [15] as well as for arbitrary three-valued expansions (cf. Corollary 4.6) of both $\mathbb{S}_3$ and HZ in [19] (cf. [16] for HZ as such). Perhaps, most acute problems remained still open concern solely the case, when K_5 forms a subalgebra of $\mathfrak{A}^2$ (in particular, when $C = NIHZ$), and are as follows:

(1) What is a relative axiomatization of the logic of $\mathcal{K}_5$?

(2) What is the lattice of those [finitary] extensions of C^{DMP}, $\mathcal{K}_5$ is a model of which?
(3) Does C^{DMP} have a strongly finite (viz., both finite itself and having merely finite members) matrix semantics and what is this (if any)?

In view of Lemma 8.3(ii),(iii)**d)**,**e)**,**g)**,**k)**(a) as well as [15] [resp., [14]], these logical problems are reduced to the following purely-algebraic ones:

(1) What is an axiomatization of the quasivariety generated by $\mathfrak{K}_5$ relatively to that generated by $\mathfrak{A}$?
(2) What is the lattice of those sub-pre[quasi]-varieties of the sub-quasi-variety $\mathsf{Q}^{\mathrm{DMP}}$ of the quasi-variety, generated by $\mathfrak{A}$, relatively axiomatized by (8.1)/(8.3)/(8.4), which contain $\mathfrak{K}_5$?
(3) Is $\mathsf{Q}^{\mathrm{DMP}}$ finitely-generated and what is a strongly finite generating set in that case?

However, these problems, entirely belonging to Universal Algebra, have appeared quite non-trivial (in particular, because of the negative result given by Lemma 8.3(iii)**k)**(c) actually leading to these problems). This is why solving them has proved beyond the scopes of the present study.

8.2. **Classically-valued implicative 3VLPSN.** Here, it is supposed that $\mathcal{A}$ is both classically-valued, and so classically-hereditary (in particular, C is $\sim$-subclassical, C^{PC} being defined by $\mathcal{A}{\restriction}2$; cf. Theorem 6.4), in which case $\sim^{\mathfrak{A}}\frac{1}{2} = 1$, while $(K|L)_{6/5}$ forms a subalgebra of $\mathfrak{A}^2$, and so $(\mathcal{K}|\mathcal{L})_{6/5} \in \mathrm{Mod}(C^{\mathrm{NP}})$, and either conjunctive or disjunctive or implicative (viz., C is so; cf. Lemmas 3.2 and 3.3) — in particular, $C = P^1$; cf. Subsubsection 4.1.3, in which case, by Remark 4.2, it is both $\bar{\wedge}$-conjunctive, $\veebar$-disjunctive, $\sqsupset$-implicative and $\neg$-negative, and so, for all $a \in A$, $\neg^{\mathfrak{A}}a = (1 - \chi^{\mathcal{A}}(a))$ and $(a \sqsupset^{\mathfrak{A}} a) = 1$.

Lemma 8.16. *For any $n \in (\omega\setminus(1(+1)))$, $A_n \triangleq (2^n \cup \{\{\langle i, \frac{1}{2}\rangle\} \cup ((n\setminus\{i\})\times\{0\}) \mid i \in n\})$ forms a subalgebra of $\mathfrak{A}^n$, $\mathcal{A}_n \triangleq (\mathcal{A}^n{\restriction}A_n) \in \mathrm{Mod}(C)$ being a consistent subdirect n-power of $\mathcal{A}$ such that ($D^{\mathcal{A}_n} = \{n \times \{1\}\}$ and) R_n is [not] true in $\mathcal{A}_{n+1[-1]}$, in which case $\mathcal{A}_{n+1} \in (\mathrm{Mod}(C_n) \setminus \mathrm{Mod}(C_{n+1}))$, and so $C_{n+1} \nsubseteq C_n$.*

Proof. Since $\mathcal{A}$ is classically-valued, the set $A_n \ni (n \times \{0\})$ does form a subalgebra of $\mathfrak{A}^n$, in which case $\mathcal{A}_n$ is consistent, for $n \neq 0$, while $D^{\mathcal{A}_n} = \{n \times \{1\}\}$, once $n \neq 1$. Then, as $\mathcal{A}$ is $\veebar$-disjunctive, R_n is not true in $\mathcal{A}_n$ under $[x_i/(\{\langle i, \frac{1}{2}\rangle\} \cup ((n \setminus \{i\}) \times \{0\})); x_n/(n \times \{0\})]_{i\in n}$ but is true in $\mathcal{A}_{n+1}$, for $\sim^{\mathfrak{A}}1 = 0$, while, for every $\bar{b} \in (\{\frac{1}{2}, 0\}^{n+1} \cap A_{n+1})^+$, $(\veebar^{\mathfrak{A}^{n+1}}\bar{b}) \in \{\frac{1}{2}, 1\}^{n+1}$ only if, for each $i \in (n+1)$, there is some $j \in (\mathrm{dom}\,\bar{b})$ such that $\pi_i(b_j) = \frac{1}{2}$ (that is, $b_j = (\{\langle i, \frac{1}{2}\rangle\} \cup (((n+1)\setminus\{i\})\times\{0\}))$) iff $(A_{n+1} \setminus 2^{n+1}) \subseteq (\mathrm{img}\,\bar{b})$, and so, for no $\bar{b} \in (\{\frac{1}{2}, 0\}^{n+1} \cap A_{n+1})^n$, $(\veebar^{\mathfrak{A}^{n+1}}\bar{b}) \in \{\frac{1}{2}, 1\}^{n+1}$, because, otherwise, we would have $(n+1) = |A_{n+1} \setminus 2^{n+1}| \leqslant |\,\mathrm{img}\,\bar{b}| \leqslant n$. □

Theorem 8.17. *$\langle C_n\rangle_{n\in\omega}$ is a strictly increasing countable chain of finitary axiomatically-equivalent (and so consistent [in particular, properly $\sim$-subclassical, that is, being proper sublogics of C^{PC}]) non-$\sim$-paraconsistent (and so proper) extensions of C, not being extensions of C^{DMP} but having model $\mathcal{K}_6$ (and so its submatrices $(\mathcal{K}|\mathcal{L})_5$), in which case they are all not extensions of C^{DN}, and so is their proper extension C_ω that is, in addition, not [relatively] finitely-axiomatizable. In particular, C_n is structurally complete, for no $n \in (\omega \setminus 1)$, while $C^{\mathrm{NP}} = C_1 \nsupseteq C_2$ is not defined by $\mathcal{K}_6 \in \mathrm{Mod}(C_2)$.*

Proof. In view of Remark 8.2 and Lemma 8.16, the ω-tuple involved is an injective (and so countable) chain, in which case C_ω is a proper extension of C_n, for any $n \in (\omega \setminus 1)$, and so, by Łoś-Mal'cev Compactness Theorem for classes of algebraic systems closed under ultra-products (cf., e.g., [7]) — in particular, finitary logic model classes, being first-order equality-free universal Horn model classes axiomatized by finitary calculi axiomatizing finitary logics, C_ω is not [relatively] finitely axiomatizable. And what is more, by Lemma 8.16, for each $n \in (\omega \setminus 1)$, $\mathcal{A}_{n+1} \in \mathrm{Mod}(C_n)$, while $(\pi_0 \restriction A_{n+1}) \in \hom^{\mathrm{S}}(\mathcal{A}_{n+1}, \mathcal{A})$, in which case $C_n \equiv_1 C$, and so $C(\varnothing) \in (\mathrm{img}\, C_n)$. Hence, $C(\varnothing) \in (\bigcap_{n \in (\omega\setminus 1)} (\mathrm{img}\, C_n)) = (\mathrm{img}\, C_\omega)$. Thus, $C_\omega \equiv_1 C$. Finally, for any $n \in (\omega \setminus 1)$, (8.1) is not true in $\mathcal{A}_{n+1} \in \mathrm{Mod}(C_n)$ (cf. Lemma 8.16) under $[x_0/(\{\langle 0, \frac{1}{2}\rangle\} \cup (((n+1)\setminus\{0\}) \times \{0\})), x_1/(\{\langle 0, 0\rangle\} \cup (((n+1)\setminus\{0\}) \times \{1\}))]$, in which case it is not satisfied in C_n, and so in C_ω, for this is the "point-wise" union of $\{C_n \mid n \in (\omega \setminus 1)\}$ (cf. Remark 8.2), as required, in view of Remark 8.1, Theorems 6.4, 7.8**a)**(viii)⇒(ii), Corollary 6.13 and Lemma 8.3(iii)(B)⇒(C). □

In view of Theorem 8.10, the implicit condition of $\mathcal{A}$'s being classically-valued cannot be omitted in the formulation of Theorem 8.17. Likewise, as opposed to Theorem 8.15(i) (in particular, Corollaries 5.4 and 8.14), by Theorem 8.17, C has infinitely many extensions, not being extensions of C^{DMP}. And what is more, in contrast to Lemma 8.9, we have:

Lemma 8.18. $\mathcal{A}_2 \in \mathrm{Mod}(C^{\mathrm{MP}}) \subseteq \mathrm{Mod}(C^{\mathrm{NP}})$ *(cf. Lemma 8.16) is a consistent subdirect square of* $\mathcal{A}$ *such that* $\hom(\mathcal{A}_2, \mathcal{A}\restriction 2) = \varnothing$.

Proof. Then, $\mathcal{B} \triangleq \mathcal{A}_2 \in \mathrm{Mod}(C)$ is a consistent subdirect square of $\mathcal{A}$. Moreover, as $2 \notin 2$, $D^{\mathcal{B}} = \{\langle 1, 1\rangle\}$, while, for every $b \in B$, it holds that $(\sim^{\mathfrak{B}} \langle 1, 1\rangle \veebar^{\mathfrak{B}} b) = (\langle 0, 0\rangle \veebar^{\mathfrak{B}} b) \in D^{\mathcal{B}}$ implies $b \in D^{\mathcal{B}}$, in view of the $\veebar$-disjunctivity of $\mathcal{A}$ and the fact that $0 \notin D^{\mathcal{A}}$. Hence, (3.2) is true in $\mathcal{B}$. Finally, let us prove, by contradiction, that $\hom(\mathcal{B}, \mathcal{A}\restriction 2) = \varnothing$. For suppose $\hom(\mathcal{B}, \mathcal{A}\restriction 2) \neq \varnothing$. Take any $h \in \hom(\mathcal{B}, \mathcal{A}\restriction 2)$, in which case $h(\langle 1, 1\rangle) = 1$, for $\langle 1, 1\rangle \in D^{\mathcal{B}}$, while $D^{\mathcal{A}\restriction 2} = \{1\}$. Therefore, if, for any $a \in \{\langle \frac{1}{2}, 0\rangle, \langle 0, \frac{1}{2}\rangle\} \subseteq B$, it did hold that $h(a) = 1$, we would have $0 = \sim^{\mathfrak{A}} 1 = h(\sim^{\mathfrak{B}} a) = h(\langle 1, 1\rangle) = 1$. Hence, $h(\langle \frac{1}{2}, 0\rangle) = 0 = h(\langle 0, \frac{1}{2}\rangle)$. Then, we get $0 = (0 \veebar^{\mathfrak{A}} 0) = h(\langle \frac{1}{2}, 0\rangle \veebar^{\mathfrak{B}} \langle 0, \frac{1}{2}\rangle) = h(\langle 1, 1\rangle) = 1$. This contradiction completes the argument. □

As a consequence, in contrast to (8.5), we get:

Corollary 8.19. $C^{\mathrm{MP}} \neq C^{\mathrm{R}}$.

Proof. By contradiction. For suppose $C^{\mathrm{MP}} = C^{\mathrm{R}}$, in which case, by Corollary 6.7 and Lemma 8.18, $\mathcal{A}_2$ is a consistent finite model of C^{PC}, defined by $\mathcal{A}\restriction 2$, such that $\hom(\mathcal{A}_2, \mathcal{A}\restriction 2) = \varnothing$, and so, by Lemma 2.2, Corollary 3.10 and Remark 4.3, there are some set I, some $\overline{\mathcal{E}} \in \mathbf{S}(\mathcal{A}\restriction 2)^I$, some subdirect product $\mathcal{F}$ of it and some isomorphism e from $\mathcal{A}_2$ onto $\mathcal{F}$. Then, $\mathcal{F}$ is consistent, for $\mathcal{A}_2$ is so, in which case $I \neq \varnothing$, and so $(e \circ \pi_i) \in \hom(\mathcal{A}_2, \mathcal{A}\restriction 2) = \varnothing$, where $i \in I \neq \varnothing$. This contradiction completes the argument. □

Finally, in contrast to Theorem 8.15(ii), we have:

Theorem 8.20. *Let* $\theta \triangleq \theta^1_{\mathfrak{A}}$ *and* $\mathcal{D} \triangleq \langle \mathfrak{Fm}^1_\Sigma, C(\varnothing) \cap \mathrm{Fm}^1_\Sigma\rangle$. *Then, the structural completion of* C *is defined by* $\mathcal{F}^1_{\mathcal{A}} \triangleq (\mathcal{D}/\theta)$, *isomorphic to* $\mathcal{K}_9 \triangleq (\mathcal{A}^3 \restriction (2^3 \cup \{\langle 0, 1, \frac{1}{2}\rangle\}))$, *an isomorphism from the former onto the latter being given by Table 1 (under identification of any* $\varphi \in \mathrm{Fm}^1_\Sigma$ *with* $\nu_\theta(\varphi)$*), and is a proper extension of the logics of* $(\mathcal{K}|\mathcal{L})_{(6/5)|5}$, *and so of their sublogics* C_α, *where* $\alpha \in (\omega + 1) \triangleq (\omega \cup \{\omega\})$, *and* $C^{(\mathrm{NP}/\mathrm{DMP})|\mathrm{DN}}$, *in which case all these proper sublogics of* C *are not structurally complete. In particular,* (8.2) *is admissible but not derivable in* C.

Proof. Then, by Lemma 8.3(iii)**b)**, $\mathfrak{A}$ is generated by the singleton $\{\frac{1}{2}\}$. Hence, by Theorem 3.11, the structural completion of C is defined by $\mathcal{F}^1_{\mathcal{A}} \triangleq (\mathcal{D}/\theta)$. Given any $a \in A$, let $h_a \in \hom(\mathfrak{Fm}^1_\Sigma, \mathfrak{A})$ extend $[x_0/a]$ and F_9 the set of all Σ-formulas appearing in the first column of Table 1. Then, as $F_9 \subseteq \mathrm{Fm}^1_\Sigma$ includes V_1 generating $\mathfrak{Fm}^1_\Sigma$, the latter is equally generated by F_9. Moreover, $h : \mathrm{Fm}^1 \to A^3, \varphi \mapsto \langle h_0(\varphi), h_1(\varphi), h_{\frac{1}{2}}(\varphi)\rangle$ is a homomorphism from $\mathfrak{Fm}^1_\Sigma$ to $\mathfrak{A}^3$ such that $h{\restriction}F_9$ is given by Table 1 (in particular, $h[F_9] = K_9$), in which case $h \in \hom(\mathfrak{Fm}^1_\Sigma, \mathfrak{K}_9)$ is surjective, for K_9 forms a subalgebra of $\mathfrak{A}^3$, because $\mathcal{A}$ is classically-valued, whereas

$$\hom(\mathfrak{Fm}^1_\Sigma, \mathfrak{A}) = \{h_a \mid a \in A\}, \tag{8.27}$$

in which case $\theta = (\bigcap_{a\in A}(\ker h_a)) = (\ker h)$, and so, by the Homomorphism Theorem, $e \triangleq (\nu_\theta^{-1} \circ h)$ is an isomorphism from $\mathfrak{F}^1_{\mathfrak{A}} = \mathfrak{F}^1_{\mathcal{A}}$ onto $\mathfrak{K}_9$. And what is more, as C is consistent, $x_0 \notin C(\varnothing) = D^{\mathcal{D}}$, in which case, for every $\varphi \in D^{\mathcal{D}}$, $h(\varphi) = \langle 1, 1, 1\rangle$, because $\mathcal{A}$ is classically-valued, and so $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{K}_9)$, for $D^{\mathcal{K}_9} = \{\langle 1, 1, 1\rangle\}$, while $1 \in D^{\mathcal{A}}$ (in particular, $h^{-1}[D^{\mathcal{K}_9}] \subseteq C(\varnothing) = D^{\mathcal{D}}$, in view of (8.27)). Thus, e is an isomorphism from $\mathcal{F}^1_{\mathcal{A}}$ onto $\mathcal{K}_9$, for $\nu_\theta \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{F}^1_{\mathcal{A}})$, in which case $h = (e \circ \nu_\theta)$, and so $F^1_{\mathcal{A}} = (F_9/\theta)$, for $h[F_9] = K_9$, while $F^1_{\mathcal{A}} = e^{-1}[K_9]$. In this way, Remark 8.1, Lemma 8.3(i)(B)⇒(C),(iii)**f**), Corollary 8.6, Theorem 8.17 and the axiomatic equivalence of the logics of $(\mathcal{K}|\mathcal{L})_{(6/5)|5}$ to C complete the argument. □

At last, in view of Theorem 8.17, P^1 has become a first instance of a three-valued conjunctive disjunctive $\sim$-subclassical $\sim$-paraconsistent Σ-logic with subclassical negation $\sim$ and infinitely many finitary extensions, despite of Theorem 8.15. On the other hand, we do not exclude that $NIHZ$ is going to become a second one, though this could not be shown by the above technique, because of Theorem 8.10 covering $NIHZ$.

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Table 1. An isomorphism from $\mathcal{F}^1_{\mathcal{A}}$ onto $\mathcal{B}$.

formula	triple
x_0	$\langle 0, 1, \frac{1}{2}\rangle$
$\sim\sim x_0$	$\langle 0, 1, 0\rangle$
$\neg x_0$	$\langle 1, 0, 0\rangle$
$\sim x_0$	$\langle 1, 0, 1\rangle$
$\neg\neg x_0$	$\langle 0, 1, 1\rangle$
$\neg\neg x_0 \barwedge \sim x_0$	$\langle 0, 0, 1\rangle$
$\sim\sim x_0 \veebar \neg x_0$	$\langle 1, 1, 0\rangle$
$x_0 \sqsupset x_0$	$\langle 1, 1, 1\rangle$
$\sim(x_0 \sqsupset x_0)$	$\langle 0, 0, 0\rangle$

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