



Properties of the Robin's Inequality

Frank Vega

EasyChair preprints are intended for rapid dissemination of research results and are integrated with the rest of EasyChair.

August 27, 2020

Properties of the Robin's Inequality

Frank Vega

Abstract. In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. The Robin's inequality consists in $\sigma(n) < e^\gamma \times n \times \ln \ln n$ where $\sigma(n)$ is the divisor function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The Robin's inequality is true for every natural number $n > 5040$ if and only if the Riemann hypothesis is true. We prove the Robin's inequality is true for every natural number $n > 5040$ when $15 \nmid n$, where $15 \nmid n$ means that n is not divisible by 15. More specifically: every counterexample should be divisible by $2^{20} \times 3^{13} \times 5^8 \times k_1$ or either $2^{20} \times 3^{13} \times k_2$ or $2^{20} \times 5^8 \times k_3$, where $15 \nmid k_1$, $3 \nmid k_2$ and $5 \nmid k_3$.

1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics [1]. It is of great interest in number theory because it implies results about the distribution of prime numbers [1]. It was proposed by Bernhard Riemann (1859), after whom it is named [1]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution [1]. The divisor function $\sigma(n)$ for a natural number n is defined as the sum of the powers of the divisors of n ,

$$\sigma(n) = \sum_{k|n} k$$

where $k | n$ means that the natural number k divides n [5]. In 1915, Ramanujan proved that under the assumption of the Riemann hypothesis, the inequality,

$$\sigma(n) < e^\gamma \times n \times \ln \ln n$$

holds for all sufficiently large n , where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant [3]. The largest known value that violates the inequality is $n = 5040$. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann hypothesis is true [3]. Using this inequality, we show an interesting result.

2010 Mathematics Subject Classification: Primary 11M26.

Keywords: number theory, inequality, divisor, prime, counterexample.

2 Results

Theorem 2.1 Given a natural number $n = p_1^{a_1} \times p_2^{a_2} \times \dots \times p_m^{a_m}$ such that p_1, p_2, \dots, p_m are prime numbers, then we obtain the following inequality

$$\frac{\sigma(n)}{n} < \prod_{i=1}^m \frac{p_i}{p_i - 1}.$$

Proof For a natural number $n = p_1^{a_1} \times p_2^{a_2} \times \dots \times p_m^{a_m}$ such that p_1, p_2, \dots, p_m are prime numbers, then we obtain the following formula

$$(2.1) \quad \sigma(n) = \prod_{i=1}^m \frac{p_i^{a_i+1} - 1}{p_i - 1}$$

from the Ramanujan's notebooks [2]. In this way, we have that

$$(2.2) \quad \frac{\sigma(n)}{n} = \prod_{i=1}^m \frac{p_i^{a_i+1} - 1}{p_i^{a_i} \times (p_i - 1)}.$$

However, for any prime power $p_i^{a_i}$, we have that

$$\frac{p_i^{a_i+1} - 1}{p_i^{a_i} \times (p_i - 1)} < \frac{p_i^{a_i+1}}{p_i^{a_i} \times (p_i - 1)} = \frac{p_i}{p_i - 1}.$$

Consequently, we obtain that

$$\frac{\sigma(n)}{n} < \prod_{i=1}^m \frac{p_i}{p_i - 1}.$$

■

Theorem 2.2 Given some prime numbers p_1, p_2, \dots, p_m , then we obtain the following inequality,

$$\prod_{i=1}^m \frac{p_i}{p_i - 1} < \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{p_i + 1}{p_i}.$$

Proof Given a prime number p_i , we obtain that

$$\frac{p_i}{p_i - 1} = \frac{p_i^2}{p_i^2 - p_i}$$

and that would be equivalent to

$$\frac{p_i^2}{p_i^2 - p_i} = \frac{p_i^2}{p_i^2 - 1 - (p_i - 1)}$$

and that is the same as

$$\frac{p_i^2}{p_i^2 - 1 - (p_i - 1)} = \frac{p_i^2}{(p_i - 1) \times \left(\frac{p_i^2 - 1}{p_i - 1} - 1\right)}$$

which is equal to

$$\frac{p_i^2}{(p_i - 1) \times \left(\frac{p_i^2 - 1}{(p_i - 1)} - 1\right)} = \frac{p_i^2}{(p_i - 1) \times \frac{p_i^2 - 1}{(p_i - 1)} \times \left(1 - \frac{(p_i - 1)}{p_i^2 - 1}\right)}$$

that is equivalent to

$$\frac{p_i^2}{(p_i - 1) \times \frac{p_i^2 - 1}{(p_i - 1)} \times \left(1 - \frac{(p_i - 1)}{p_i^2 - 1}\right)} = \frac{p_i^2}{p_i^2 - 1} \times \frac{1}{1 - \frac{(p_i - 1)}{p_i^2 - 1}}$$

which is the same as

$$\frac{p_i^2}{p_i^2 - 1} \times \frac{1}{1 - \frac{(p_i - 1)}{p_i^2 - 1}} = \frac{1}{1 - p_i^{-2}} \times \frac{1}{1 - \frac{1}{(p_i + 1)}}$$

and finally

$$\frac{1}{(1 - p_i^{-2})} \times \frac{1}{1 - \frac{1}{(p_i + 1)}} = \frac{1}{(1 - p_i^{-2})} \times \frac{p_i + 1}{p_i}.$$

In this way, we have that

$$\prod_{i=1}^m \frac{p_i}{p_i - 1} = \prod_{i=1}^m \frac{1}{1 - p_i^{-2}} \times \prod_{i=1}^m \frac{p_i + 1}{p_i}.$$

However, we know that

$$\prod_{i=1}^m \frac{1}{1 - p_i^{-2}} < \prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-2}}$$

where p_j is the j^{th} prime number and we have that

$$\prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-2}} = \frac{\pi^2}{6}$$

as a consequence of the result in the Basel problem [5]. Consequently, we obtain that

$$\prod_{i=1}^m \frac{p_i}{p_i - 1} < \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{p_i + 1}{p_i}.$$

■

Definition 2.3 We recall that an integer n is said to be squarefree if for every prime divisor p of n we have $p^2 \nmid n$, where $p^2 \nmid n$ means that p^2 does not divide n [3].

Theorem 2.4 Given a squarefree number $n = q_1 \times \dots \times q_m$ such that q_1, q_2, \dots, q_m are odd prime numbers, $3 \nmid n$ and $5 \nmid n$, then we obtain the following inequality

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^\gamma \times n \times \ln \ln(2^{19} \times n).$$

Proof This proof is very similar with the demonstration in Theorem 1.1 from the article reference [3]. By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [3]. Put $\omega(n) = m$ [3]. We need to prove the assertion for those integers with $m = 1$. From the equation (2.1), we obtain that

$$(2.3) \quad \sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \dots \times (q_m + 1)$$

when $n = q_1 \times q_2 \times \dots \times q_m$. In this way, for any prime number $p_i \geq 7$, then we need to prove

$$(2.4) \quad \frac{\pi^2}{6} \times \frac{3}{2} \times \left(1 + \frac{1}{p_i}\right) \leq e^\gamma \times \ln \ln(2^{19} \times p_i).$$

For $p_i = 7$, we have that

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \left(1 + \frac{1}{7}\right) \leq e^\gamma \times \ln \ln(2^{19} \times 7)$$

is actually true. For another prime number $p_i > 7$, we have that

$$\left(1 + \frac{1}{p_i}\right) < \left(1 + \frac{1}{7}\right)$$

and

$$e^\gamma \times \ln \ln(2^{19} \times 7) < e^\gamma \times \ln \ln(2^{19} \times p_i)$$

which clearly implies that the inequality (2.4) is true for every prime number $p_i \geq 7$. Now, suppose it is true for $m - 1$, with $m \geq 1$ and let us consider the assertion for those squarefree n with $\omega(n) = m$ [3]. So let $n = q_1 \times \dots \times q_m$ be a squarefree number and assume that $q_1 < \dots < q_m$ for $q_m \geq 7$.

Case 1 : $q_m \geq \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m) = \ln(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \dots \times (q_{m-1} + 1) \leq e^\gamma \times q_1 \times \dots \times q_{m-1} \times \ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \dots \times (q_{m-1} + 1) \times (q_m + 1) \leq$$

$$e^\gamma \times q_1 \times \dots \times q_{m-1} \times (q_m + 1) \times \ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1})$$

when we multiply the both sides of the inequality by $(q_m + 1)$. We want to show that

$$e^\gamma \times q_1 \times \dots \times q_{m-1} \times (q_m + 1) \times \ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1}) \leq$$

$$e^\gamma \times q_1 \times \dots \times q_{m-1} \times q_m \times \ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m) = e^\gamma \times n \times \ln \ln(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m) - \ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1}))}{\ln q_m} \geq$$

$$\frac{\ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1})}{\ln q_m}.$$

From the reference [3], we have that if $0 < a < b$, then

$$(2.5) \quad \frac{\ln b - \ln a}{b - a} = \frac{1}{(b - a)} \int_a^b \frac{dt}{t} > \frac{1}{b}.$$

We can apply the inequality (2.5) to the previous one just using $b = \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m)$ and $a = \ln(2^{19} \times q_1 \times \dots \times q_{m-1})$. Certainly, we have that

$$\begin{aligned} & \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m) - \ln(2^{19} \times q_1 \times \dots \times q_{m-1}) = \\ & \ln \frac{2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \dots \times q_{m-1}} = \ln q_m. \end{aligned}$$

In this way, we obtain that

$$\begin{aligned} & \frac{q_m \times (\ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m) - \ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1}))}{\ln q_m} > \\ & \frac{q_m}{\ln(2^{19} \times q_1 \times \dots \times q_m)}. \end{aligned}$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\ln(2^{19} \times q_1 \times \dots \times q_m)} \geq \frac{\ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1})}{\ln q_m}$$

which is trivially true for $q_m \geq \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m)$ [3].

Case 2 : $q_m < \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m) = \ln(2^{19} \times n)$.

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^\gamma \times \ln \ln(2^{19} \times n).$$

We know that $\frac{3}{2} < 1.6 = \frac{4 \times 6}{3 \times 5}$. Nevertheless, we could have that

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times 6 \times \sigma(n)}{3 \times 5 \times n} \times \frac{\pi^2}{6} = \frac{\sigma(3 \times 5 \times n)}{3 \times 5 \times n} \times \frac{\pi^2}{6} \leq e^\gamma \times \ln \ln(2^{19} \times n)$$

where this is possible because of $3 \nmid n$ and $5 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain that

$$\begin{aligned} & \ln\left(\frac{\pi^2}{6}\right) + (\ln(3+1) - \ln 3) + (\ln(5+1) - \ln 5) + \sum_{j=3}^m (\ln(q_j+1) - \ln q_j) \leq \\ & \gamma + \ln \ln \ln(2^{19} \times n). \end{aligned}$$

From the reference [3], we note that

$$\ln(p_1 + 1) - \ln p_1 = \int_{p_1}^{p_1+1} \frac{dt}{t} < \frac{1}{p_1}.$$

In addition, note also that $\ln(\frac{\pi^2}{6}) < \frac{1}{2}$. It is enough to prove that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{q_1} + \dots + \frac{1}{q_m} \leq \sum_{p \leq q_m} \frac{1}{p} + \leq \gamma + \ln \ln \ln(2^{19} \times n)$$

where $p \leq q_m$ means all the prime lesser than or equal to q_m . However, we know that

$$\gamma + \ln \ln q_m < \gamma + \ln \ln \ln(2^{19} \times n)$$

since $q_m < \ln(2^{19} \times n)$ and therefore, we would only need to prove that

$$\sum_{p \leq q_m} \frac{1}{p} \leq \gamma + \ln \ln q_m$$

which is true according to the Lemma 2.1 from the article reference [3]. In this way, we finally show the Theorem is indeed satisfied. ■

Theorem 2.5 *Given a natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$ such that $a_1, a_2, a_3 \geq 0$ are integers, then the Robin's inequality is true for n .*

Proof Given a natural number $n = p_1^{a_1} \times p_2^{a_2} \times \dots \times p_m^{a_m} > 5040$ such that p_1, p_2, \dots, p_m are prime numbers, we need to prove that

$$\frac{\sigma(n)}{n} < e^\gamma \times \ln \ln n$$

that would be the same as

$$(2.6) \quad \prod_{i=1}^m \frac{p_i}{p_i - 1} < e^\gamma \times \ln \ln n$$

according to Theorem 2.1. Given a natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$ such that $a_1, a_2, a_3 \geq 0$ are integers, we have that

$$\prod_{i=1}^m \frac{p_i}{p_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^\gamma \times \ln \ln(5040) \approx 3.81.$$

However, we know for $n > 5040$, we have that

$$e^\gamma \times \ln \ln(5040) < e^\gamma \times \ln \ln n$$

and thus, the proof is completed. ■

Theorem 2.6 *The Robin's inequality is true for every natural number $n > 5040$ when $15 \nmid n$. More specifically: every counterexample should be divisible by $2^{20} \times 3^{13} \times 5^8 \times k_1$ or either $2^{20} \times 3^{13} \times k_2$ or $2^{20} \times 5^8 \times k_3$, where $15 \nmid k_1$, $3 \nmid k_2$ and $5 \nmid k_3$.*

Proof Given a natural number $n = p_1^{a_1} \times p_2^{a_2} \times \dots \times p_m^{a_m} > 5040$ such that p_1, p_2, \dots, p_m are prime numbers, then we will check the Robin's inequality for n . We know this true when the greatest prime divisor of n is lesser than or equal to 5 according to Theorem 2.5. Another case is when $3 \nmid n$ and $5 \nmid n$.

We need to prove the inequality (2.6) for that case. In addition, the inequality (2.6) would be true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{p_i + 1}{p_i} < e^\gamma \times \ln \ln n$$

according to Theorem 2.2. Using the properties of the equation (2.2), we obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} < e^\gamma \times \ln \ln n$$

where $n' = q_1 \times \dots \times q_m$ is the squarefree representation of n . However, the Robin's inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [3]. Hence, we need to prove when $2 \mid n'$. In addition, we know the Robin's inequality is true for every $n > 5040$ such that $2^k \mid n$ for $1 \leq k \leq 19$ [4]. Consequently, we only need to prove that for all $n > 5040$ such that $2^{20} \mid n$ and thus, we have that

$$e^\gamma \times n' \times \ln \ln(2^{19} \times \frac{n'}{2}) < e^\gamma \times n' \times \ln \ln n$$

because of $2^{19} \times \frac{n'}{2} < n$ when $2^{20} \mid n$ and $2 \mid n'$. In this way, we only need to prove that

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^\gamma \times n' \times \ln \ln(2^{19} \times \frac{n'}{2}).$$

According to the equation (2.3) and $2 \mid n'$, we have that

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^\gamma \times 2 \times \frac{n'}{2} \times \ln \ln(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^\gamma \times \frac{n'}{2} \times \ln \ln(2^{19} \times \frac{n'}{2})$$

which is true according to the Theorem 2.4. In addition, we know the Robin's inequality is true for every $n > 5040$ such that $3^i \mid n$ and $5^j \mid n$ for $1 \leq i \leq 12$ and $1 \leq j \leq 7$ [4]. To sum up, we have finally proved this result. ■

References

- [1] The millennium problems: the seven greatest unsolved mathematical puzzles of our time. *Choice Reviews Online*, 41(02):41–0977, oct 2003. URL: <https://doi.org/10.5860%2Fchoice.41-0977>, doi:10.5860/choice.41-0977.
- [2] Bruce C Berndt. *Ramanujans notebooks: Part III*. Springer Science & Business Media, 2012.
- [3] YoungJu Choie, Nicolas Lichiardopol, Pieter Moree, and Patrick Solé. On Robin's criterion for the Riemann hypothesis. *Journal de Théorie des Nombres de Bordeaux*, 19(2):357–372, 2007. URL: <https://doi.org/10.5802%2Fjtnb.591>, doi:10.5802/jtnb.591.
- [4] Alexander Hertlein. Robin's inequality for new families of integers. *arXiv preprint arXiv:1612.05186*, 2016.
- [5] David G. Wells. *Prime Numbers, The Most Mysterious Figures in Math*. John Wiley & Sons, Inc., 2005.

CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France
Email: vega.frank@gmail.com